Infinitesimal remark on Tsimerman's proof of André-Oort for A_g.

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Abstract.

We prove Tsimerman's Galois orbit lower bound for CM points on A_g.

1 Introduction.

Theorem 1.1 (Tsimerman [7]). *Let $g \in \mathbb{Z}^+$. Let $K/\mathbb{Q}$ be a number field. Let $L/\mathbb{Q}$ be a CM $\mathbb{Q}$-algebra of dimension $\dim_{\mathbb{Q}} L = 2g$. Let A/K be a g-dimensional abelian variety admitting $L \simeq End^0_K(A)$. Then:*

$$[K : \mathbb{Q}] \gg_g \operatorname{disc}(L)^{\Omega_g(1)}.$$

2 Endomorphism estimate.

Theorem 2.1 ([coarse version of] Masser-Wüstholz). *Let $g \in \mathbb{Z}^+$. Let $K/\mathbb{Q}$ be a number field. Let $A, B/K$ with $A \sim_K B$ be g-dimensional abelian varieties. Then: there is a K-isogeny $\varphi : A \longrightarrow B$ with*

$$\deg \varphi \leq \max \left(10^{10} \cdot [K : \mathbb{Q}], h(A)\right)^{O_g(1)}.$$

Tsimerman applies Theorem 2.1 to every B/K with $B \sim_K A$ when A/K is CM. Because[1] the isogeny class of A/K is of size polynomial in $\operatorname{disc}(L)$, and because[2] $h(A)$ is subpolynomial in $\operatorname{disc}(L)$, Theorem 1.1 follows.

But transcendence techniques give more control than on just the minimal degree isogeny. The following argument applies to similarly produce a $\mathbb{Z}$-basis of $\operatorname{Hom}_K(A, B)$ of K-isogenies of degree at most the same bound for *all* g-dimensional $A, B/K$ with $A \sim_K B$, but we will state the theorem in the CM case to avoid pointless notation.

[1](by CM theory [4] and the usual class number lower bounds, some of which are effective thanks to Stark's [6])

[2](by the proof of the average Colmez conjecture [1, 8], Bost's lower bound on Faltings heights [3], and the usual ineffective sharp class number lower bounds [5])

Theorem 2.2 (immediate corollary of a bound of Gaudron-Rémond's)**.** *Let* $g \in \mathbb{Z}^+$*. Let* $K/\mathbb{Q}$ *be a number field. Let* $L/\mathbb{Q}$ *be a CM* $\mathbb{Q}$*-algebra of dimension* $\dim_{\mathbb{Q}} L = 2g$*. Let* A/K *be a* g*-dimensional abelian variety Then:*

$$\mathrm{disc}(L) \leq \max\left(10^{10} \cdot [K:\mathbb{Q}], h(A)\right)^{O_g(1)}.$$

Proof. This follows from Lemma 9.2 of Gaudron-Rémond's [2] because (in their notation) $v(A) \ll_g 1$, their $\Lambda = \Lambda(Z(A), Z(A)) \geq \mathrm{vol}(\mathrm{End}_K(Z(A)))^{\Omega_g(1)}$ by Hadamard, the latter is[3] $\geq \mathrm{vol}(\mathrm{End}_K(A))^{\Omega_g(1)}$, and, as they show in the proof of their Proposition 2.9, $\mathrm{vol}(\mathrm{End}_K(A)) = \mathrm{disc}(L)$ (by writing the left-hand side as the usual definition of the right-hand side). □

3 Proof of Theorem 1.1.

Proof. Apply Theorem 2.2 and conclude in the same way: by Brauer-Siegel, the average Colmez conjecture, and Bost, one has[4] $h(A) \ll \mathrm{disc}(L)^{o(1)}$. □

References.

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[7] Jacob Tsimerman. The André-Oort conjecture for $\mathcal{A}_g$. *Ann. of Math. (2)*, 187(2):379–390, 2018.

[8] Xinyi Yuan and Shou-Wu Zhang. On the averaged Colmez conjecture. *Ann. of Math. (2)*, 187(2):533–638, 2018.

[3](— see the remark after their Corollary 2.10.)

[4](an ineffective bound, thanks very much to Siegel zeroes — though by Tatuzawa only ineffective for those L containing one particular imaginary quadratic field)