The $(3, 4, \infty)$ modular family of 2-tori, completed at its three special points, is a complex structure on S^6.

The base and the family. Let $\Delta = \Delta(3, 4, \infty)$ be the triangle group, acting on the upper half plane $\mathfrak{h}$; the quotient $\mathfrak{h}/\Delta$ is $\mathbb{P}^1$ minus a point, and adding the cusp gives $\mathbb{P}^1$ with orbifold points of orders 3 and 4 and the cusp. Let $V \cong \mathbb{Z}^4$ carry automorphisms T_1, T_2 of orders 3 and 4 with $T_0 := (T_1T_2)^{-1}$ unipotent, $(T_0 - I)^2 = 0$: this is a representation $\Delta \to \mathrm{SL}(V)$, and we let it act on the dual lattice $\Lambda = V^*$ by $A_j = (T_j^{-1})^t$ (§2). The period functions τ, μ, β on $\mathfrak{h}$ are an equivariant period map for it (§3), and the quotient is a family of complex 2-tori over the punctured orbifold curve: the analogue of a universal family of abelian surfaces over a modular curve, with this rank-four lattice representation in place of the symplectic one. The very general fibre is not algebraic. The very general fibre F has $\mathrm{NS}(F) = \mathbb{Z}\eta$ with η a single indefinite class, and the algebraic dimension of the completed threefold is $a(X) = 1$ (§9).

The three special fibres. The three conjugacy classes of maximal finite and parabolic cyclic subgroups of Δ single out the three special points, and at these three points we take the three standard completions; each is a choice, and every ℓ_0 and every admissible v_j below occurs. At an elliptic point of order m the local monodromy is a torus automorphism of order m, and the filling is Kodaira's logarithmic transform: $(\text{disc} \times \text{torus})/(\mathbb{Z}/m)$, the generator acting by a rotation of the disc and by the automorphism followed by a translation; the reduced fibre is of bielliptic type (§5). At the cusp the monodromy is unipotent of rank two, and the filling N_0 is Mumford's toric degeneration; its central fibre W is read off the prescribed A_2-triangulation of $\mathbb{R}^2$, the deck action being the one in which the unimodular map $B_0 \colon \bar{\Lambda} \to \Lambda_{\mathrm{tor}}$ induced by $M_0 - I = (T_0^{-1})^t - I$ on Λ enters. Modulo the lattice the triangulation has one vertex, three edges and two triangles, i.e. W is one toric surface with hexagonal moment polygon — the degree-six del Pezzo dP_6 — glued to itself along opposite sides, with two triple points (§4). Thus there is a single family, completed in the three standard ways at the three kinds of special point of an orbifold curve with a cusp.

The gluing and the fundamental group. Each filling is glued to the global family along a collar by a fibre-preserving map, and we reglue by translations by local sections (§6, Prop. 6.3); among these, π_1 and H_* depend only on three integers ℓ_0 (at the cusp), ℓ_1, ℓ_2 (at the elliptic points) (§7). Let $\gamma \in V$ be the linear coordinate on Λ whose kernel $\ker\gamma = \langle \hat{u}, \hat{w}, \hat{\delta} \rangle$ is the span of all $(A_j - 1)\Lambda$; then the coinvariants $\Lambda/\ker\gamma \cong \mathbb{Z}$ are detected by γ alone, and $\ell_j = \gamma(v_j)$ for the translation vectors v_j. The result is $\pi_1(X) \cong \mathbb{Z}/|12\ell_0 - 4\ell_1 - 3\ell_2|$ (§7).[1] This formula admits a reading, which the body does not use. It is the formula $|a_1a_2e_0 + a_2b_1 + a_1b_2|$ of [Orl72] for the order of H_1 of the Seifert fibred space over $S^2(a_1, a_2)$, evaluated at $e_0 = \ell_0$ and $b_j = -\ell_j$ — the sign coming from the convention $s_j \circ g_j = e^{-2\pi i/m_j} s_j$ for the local sections — so here $(a_1, a_2) = (3, 4)$ with Seifert invariants $(3, -\ell_1)$, $(4, -\ell_2)$ and obstruction ℓ_0; for $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$ the value is 1 and the space in question is S^3 with the circle action $(z, w) \mapsto (\lambda^3 z, \lambda^4 w)$, whose orbit space is $S^2(3, 4)$ [Orl72]. Heuristically the twist condition says: glue so that the coinvariant circle of the fibres traces out the $(3, 4)$ Seifert fibration of the three-sphere over the base.

The homology and the Euler number. The remaining torus directions contribute nothing: $\hat{w}$ and $\hat{\delta}$ are vanishing cycles at the cusp, collapsed in the hexagonal fibre, and the rest is killed by monodromy, the coinvariants of Λ having rank one. With the twists above, X has the integral homology of S^6 and is diffeomorphic to S^6 (§7, §8). The Euler number localises at the cusp: $e(X) = e(W) = 2$ (§7), every other fibre being a torus or a free torus quotient and so of Euler number 0 — one singular fibre carries the whole Euler characteristic of S^6, as the twelve nodal fibres carry the 12 of a rational elliptic surface.

Why the argument of [CDP20] does not apply. That argument requires a line bundle non-torsion on every fibre and propagates a vanishing statement through each singular fibre by way of its normalisation. Here $H^2(X; \mathbb{Z}) = 0$ and $\mathrm{Pic}(X) \cong \mathbb{C}$, so every line bundle is topologically trivial; at the non-normal toric fibre the differential of the fibration itself supplies the section which the passage to the normalisation assumes away, and $R^2 f_*(T_X \otimes L) \neq 0$ for all L. The count of H_1 in [CDP20, Lemma 4.2] (after [CDP98, Lemma 3.2]) assumes trivial monodromy, whereas the monodromy representation here is non-trivial; §10 shows that this last point is repairable for X, while the failure of the reduction at the non-normal fibre is decisive.

[1] §7 proves $\pi_1(X) \cong \mathbb{Z}/|p|$, $p = 12\ell_0 - 4\ell_1 - 3\ell_2$, and computes $H_*(X; \mathbb{Z})$ for every admissible pair (v_1, v_2); the value $|p| = 1$ is one point of that family.

Setup. $V := \mathbb{Z}^4$, basis (γ, u, w, δ); $\Lambda := V^*$, dual $(\hat\gamma, \hat u, \hat w, \hat\delta)$; $T_1, T_2 \in \mathrm{Aut}(V)$, orders $3,4$ (columns: images), and $\Pi(z)$ $(\tau,\mu,\beta$ below):

$$T_1 = \begin{pmatrix} 1 & 0 & -6 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 6 & 0 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Pi(z) := \begin{pmatrix} 6\mu & \tau & 1 & 0 \\ \beta & \mu & 0 & 1 \end{pmatrix}$$

$T_0 := (T_1T_2)^{-1} = I + N$, $N^2 = 0$, $N\gamma = Nu = 0$, $Nw = -u$, $N\delta = \gamma$; on Λ: $A_j := (T_j^{-1})^t$, $M_0 := (T_0^{-1})^t$, $\Lambda_{\mathrm{tor}} := \langle \hat w, \hat\delta\rangle$, $\overline{\Lambda} := \Lambda/\Lambda_{\mathrm{tor}}$, $B_0 \colon \overline{\Lambda} \xrightarrow{\sim} \Lambda_{\mathrm{tor}}$, $\bar{\hat\gamma} \mapsto -\hat\delta$, $\bar{\hat u} \mapsto \hat w$; A_1, A_2 fix $\varepsilon := \hat\gamma + 2\hat u - 4\hat w$, $\varepsilon' := \hat\gamma + 3\hat u - 3\hat w$ resp. $B := \mathbb{P}^1 \ni t$, $t_c := 1/t$, $p_1, p_2, p_0 := 0, 1, \infty$, $B^\circ := B \setminus \{p_0, p_1, p_2\}$, $m_1, m_2 := 3, 4$; $\pi \colon \mathfrak{h}_z \to B \setminus \{p_0\}$ ($\mathfrak{h}_z, \mathfrak{h}$ upper half-pl.) orbifold uniformisation (m_j at p_j, cusp p_0), deck gp. $\Delta = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1\rangle$, $g_j z_j = z_j \in \pi^{-1}(p_j)$, $g_j'(z_j) = e^{-2\pi i/m_j}$, $g_0 := (g_1g_2)^{-1}$ parabolic; $\rho_V \colon \Delta \to GL(V)$, $g_j \mapsto T_j$, $M_g := \rho_V(g)^t$; $U_r \subset \pi^{-1}\{0 < |t_c| < r\}$ (r small) $\langle g_0\rangle$-inv. cusp comp.; hol. $\tau \colon \mathfrak{h}_z \to \mathfrak{h}$, $\mu, \beta \colon \mathfrak{h}_z \to \mathbb{C}$ (j modular invariant, $j(i) = 1728$):

- **(τ1)**: $j(\tau) = 1728\, t \circ \pi$, **($\tau$2)**: $\tau \circ g_1 = \frac{\tau - 1}{\tau}$, $\tau \circ g_2 = -\frac{1}{\tau}$,
- **(τ3)**: $\tau(z_1) = e^{i\pi/3}$, $\tau(z_2) = i$, **(μ1)**: $\mu \circ g_1 = \frac{1-\mu}{\tau}$, $\mu \circ g_2 = 1 + \frac{\mu}{\tau}$,
- **(μ2)**: μ bounded on U_r, **(β1)**: $\beta \circ g_1 = \beta + 2 - \frac{6(1-\mu)^2}{\tau}$, $\beta \circ g_2 = \beta - 3 - \frac{6\mu^2}{\tau}$,
- **(β2)**: $\beta + \tau$ bounded on U_r, **(β3)**: $\operatorname{Im}\beta - \frac{6(\operatorname{Im}\mu)^2}{\operatorname{Im}\tau} < 0$ on $\mathfrak{h}_z$.

Such τ, μ, β exist by Theorem 3.4; on $\mathbb{C}^2 \ni \zeta = (\zeta_1, \zeta_2)$, $\Pi(z)$ as above with columns $\Pi(z)\hat\gamma, \Pi(z)\hat u, \Pi(z)\hat w, \Pi(z)\hat\delta$: $\lambda \cdot (z, \zeta) := (z, \zeta + \Pi(z)\lambda)$, unique $R_g(z) \in GL_2(\mathbb{C})$: $\Pi(gz) = R_g(z)\Pi(z)M_g$, $\mathcal{T} := (\mathfrak{h}_z \times \mathbb{C}^2)/\Lambda$, Δ by $\tilde g(z, \zeta) := (gz, R_g(z)\zeta)$, $J := (\mathcal{T}|_{\mathfrak{h}_z \setminus \Delta\{z_1, z_2\}})/\Delta \to B^\circ$, monodromy A_1, A_2, M_0 on $H_1 = \Lambda$. On U_r, $e^{2\pi i \tau} = t_c u_1(t_c)$, u_1 unit at 0; for $s := \tau - (2\pi i)^{-1}\log u_1$ (any branch; §4) $\Pi = [sB_0 + C(t_c) \mid I]$ in bases $(\bar{\hat\gamma}, \bar{\hat u})$, $(\hat w, \hat\delta)$, C hol. at 0. $N' := \mathbb{Z}^3 \ni (y, y_3)$, $y \in \mathbb{Z}^2 = \Lambda_{\mathrm{tor}}$ ($e_1 \leftrightarrow \hat w$, $e_2 \leftrightarrow \hat\delta$), cochar. lattice of $(\mathbb{C}^*)^3 \ni (x_1, x_2, t_c) \subset Y$ smooth toric, fan $\mathfrak{F}$ cone over A_2-triangulation of $\mathbb{R}^2 \times \{1\}$ (vertices $\mathbb{Z}^2 \times \{1\}$, edges $\pm e_1, \pm e_2, \pm(e_1 - e_2)$), $x_k = e^{2\pi i \zeta_k}$, t_c character of $(0,0,1) \in \mathrm{Hom}(N', \mathbb{Z})$, hol. on Y. $\overline{\Lambda}$ acts freely, prop. disc. on $Y_\epsilon := \{|t_c| < \epsilon\}$, ϵ small (4.5), by $\Psi_{\bar\lambda} := (\exp 2\pi i\, C(t_c)\bar\lambda, 1) \cdot \Phi_{\bar\lambda}$, $\Phi_{\bar\lambda}$ from $(y, y_3) \mapsto (y + y_3 B_0\bar\lambda, y_3)$; $N_0 := Y_\epsilon/\overline{\Lambda}$ **(N_0)**, $W := \{t_c = 0\}/\overline{\Lambda}$, $\eta \colon \mathrm{dP}_6 \to W$ normalisation, dP_6 del Pezzo of degree 6. For $j = 1, 2$: $\Delta_j \ni z_j$ g_j-inv. disc, coord. s_j, $s_j \circ g_j = e^{-2\pi i/m_j} s_j$, $s_j^{m_j} = t_j \circ \pi$, t_j coord. at p_j, $N_j := (\mathcal{T}|_{\Delta_j})/\langle (z, \zeta) \mapsto (g_j z, R_{g_j}(z)\zeta + \Pi(g_j z)v_j/m_j)\rangle$ **(N_1)**, **(N_2)**, $v_1 := \varepsilon$, $v_2 := -\varepsilon'$ (free; 5.4). Glue to J, any branch: N_j by $\zeta \mapsto \zeta + (2\pi i)^{-1}(\log s_j)\Pi(z)v_j$ (§5), $(Y_\epsilon \cap (\mathbb{C}^*)^3)/\overline{\Lambda}$ on $0 < |t_c| < \epsilon$ by $(z, \zeta) \mapsto (e^{2\pi i \zeta_1}, e^{2\pi i \zeta_2}, e^{2\pi i s(z)})$. $\ell_j := \gamma(v_j)$ ($\gamma \in V = \mathrm{Hom}(\Lambda, \mathbb{Z})$): $\ell_1 = 1$, $\ell_2 = -1$, $\ell_0 := 0$. $X := (J \sqcup N_0 \sqcup N_1 \sqcup N_2)/\sim$ (both gluings), $f \colon X \to B$ induced.

Theorem. *X is a compact connected complex 3-manifold and $f \colon X \to B$ is a surjective holomorphic map with connected fibres.*

1. *Over B° the map f is a proper submersion whose fibres are complex 2-tori. The fibre $f^{-1}(p_0)$ is the divisor $W \subset N_0$, which is reduced, irreducible and has normal crossings. The normalisation η identifies opposite sides of the hexagon of (-1)-curves of dP_6, the double locus is three rational curves through two triple points, and $e(W) = 2$. For $j = 1, 2$ one has $f^*(p_j) = m_j S_j$, where $S_j := (f^{-1}(p_j))_{\mathrm{red}}$ is a smooth bielliptic surface in N_j, and $\mathcal{O}_X(S_j)|_{S_j}$ has order m_j.*
2. *The threefold X is simply connected, since $\pi_1(X) \cong \mathbb{Z}/|12\ell_0 - 4\ell_1 - 3\ell_2| = 1$, and $H_*(X; \mathbb{Z}) \cong H_*(S^6; \mathbb{Z})$; consequently X is diffeomorphic to S^6 (§7ff.).*
3. *The algebraic dimension of X is $a(X) = 1$ and $\mathcal{M}(X) = f^*\mathbb{C}(t)$: every meromorphic function on X is constant along the fibres of f.*
4. *The canonical bundle is $K_X \cong f^*\mathcal{O}_B(-1) \otimes \mathcal{O}_X(2S_2)$ and is not torsion; $c_3(X) = 2$; and $\mathrm{Aut}^0(X) \cong \mathbb{C}^*$ acts along the fibres of f with fixed locus a double curve of W (§9).*

Remark (on [CDP20]). The Theorem contradicts [CDP20, Cor. 2.3] as published. For $L \in \mathrm{Pic}(X)$ set $A := (L^* \otimes K_X)|_W$: η^*A is trivial ($H^2(X; \mathbb{Z}) = 0$, $\mathrm{Pic}(\mathrm{dP}_6)$ discrete), and for θ trivialising it $d(t_c \circ f)|_W \otimes \theta$ vanishes on the double locus, so descends to $0 \neq \sigma \in \Omega^1_X|_W \otimes A$ dying in $\widetilde\Omega^1_W \otimes A$ ($\widetilde\Omega := \Omega/\mathrm{torsion}$). By Serre–Grothendieck duality on W ($\omega_W \cong K_X|_W$) and base change $R^2 f_*(T_X \otimes L) \neq 0$ for all L (§10); so hypothesis (1) of [CDP20, Prop. 2.4, p. 680] holds for no L. The reduction of [CDP20, p. 692] to $H^0(\widetilde\Omega^1_S \otimes \cdot) = 0$ on reduced fibres S is stated there provided $L|_S$ is non-torsion, a proviso met at W for general L, where the conclusion $H^0(\widetilde\Omega^1_W \otimes A) = 0$ does hold; but it does not suffice. Indeed, for general L, $A \not\cong \mathcal{O}_W$ forces $H^0(W, N^*_W \otimes A) = H^0(W, A) = 0$, so the conormal term vanishes identically, while σ gives $H^0(W, \Omega^1_X|_W \otimes A) \neq 0$: this is where the reduction fails.

A compact complex threefold fibred by tori over the projective line, and the six-sphere.

Abstract

We construct a compact connected complex manifold X of dimension three together with a surjective holomorphic map $f\colon X \to \mathbb{P}^1$ whose fibres over the complement of three points p_0, p_1, p_2 are complex tori of dimension two, whose fibre over p_0 is a reduced irreducible rational surface with normal crossings (a del Pezzo surface of degree six with the opposite sides of its anticanonical hexagon identified), and whose fibres over p_1 and p_2 are multiple, of multiplicities 3 and 4, with bielliptic reductions S_1, S_2. The family over the thrice punctured sphere is given by explicit period functions on the upper half plane attached to the $(3, 4, \infty)$ orbifold; the fibre over p_0 is filled in by a quotient of an infinite smooth toric threefold, and the fibres over p_1, p_2 by logarithmic transforms. We prove that X is simply connected with the integral homology of S^6, hence diffeomorphic to S^6, and we compute $a(X) = 1$ with f the algebraic reduction — the Néron–Severi group of the very general fibre being infinite cyclic, generated by a class η whose associated hermitian form H_η has signature $(1, 1)$ and which is therefore represented by no positive line bundle, so that the very general fibre has algebraic dimension 0 — together with $f_*\mathcal{O}_X = \mathcal{O}$, $R^1 f_*\mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-1)$, $R^2 f_*\mathcal{O}_X \cong \mathcal{O}(-1)$, $h^{0,1} = 1$, $h^{0,2} = h^{0,3} = 0$, $h^{1,0} = h^{2,0} = 0$, $K_X \cong f^*\mathcal{O}(-1) \otimes \mathcal{O}_X(2S_2)$, $c_1c_2 = 0$, $c_3 = 2$, $h^0(T_X) = 1$ and $\mathrm{Aut}^0(X) \cong \mathbb{C}^*$. The existence of such an X is incompatible with [CDP20, Cor. 2.3]; the last section locates the point at which the two accounts diverge, by showing that $R^2 f_*(T_X \otimes L) \neq 0$ for every holomorphic line bundle L on X, a consequence of the non-normality of the normal crossings fibre W.

Contents

1 Introduction.

1.1 The result

Throughout, $B := \mathbb{P}^1$ carries the affine coordinate t, and p_1, p_2, p_0 are the points $t = 0$, $t = 1$, $t = \infty$, with $t_c := 1/t$ the coordinate at p_0 and $B^\circ := B \setminus \Sigma$, $\Sigma := \{p_0, p_1, p_2\}$; further $m_1 := 3$, $m_2 := 4$. We write dP_6 for the del Pezzo surface of degree 6, the blow up of $\mathbb{P}^2$ in three non-collinear points, whose anticanonical cycle is a hexagon of six (-1)-curves. A *bielliptic* surface is a smooth compact complex surface $(E \times F)/G$ with E, F elliptic curves and G a finite group of translations of E acting on F with $F/G \cong \mathbb{P}^1$ [BHPV, V.5], [BdF], [Ser]. We write $\mathcal{M}(Y)$ for the field of meromorphic functions of a compact complex space Y, and $a(Y) = \operatorname{tr.deg}_{\mathbb{C}} \mathcal{M}(Y)$ for its algebraic dimension.

Theorem. There exist a compact connected complex manifold X of dimension 3 and a surjective holomorphic map $f\colon X \to B = \mathbb{P}^1$ with connected fibres, with the following properties.

(1) f is a proper holomorphic submersion over B°, and every fibre $f^{-1}(p)$ with $p \in B^\circ$ is a complex torus of dimension 2.

(2) The fibre $W := f^{-1}(p_0)$ is reduced, irreducible, and a normal crossings divisor in X: at every point of W the function $t_c \circ f$ is, in suitable local coordinates, z_1, $z_1 z_2$ or $z_1 z_2 z_3$. The normalisation of W is dP_6, and W is obtained from dP_6 by identifying the three pairs of opposite sides of its hexagon of (-1)-curves. The double locus $\mathsf{D} \subset W$ consists of three smooth rational curves, each passing through both triple points of W and otherwise pairwise disjoint; and $e(W) = 2$.

(3) The fibres over p_1 and p_2 are multiple: as divisors, $f^*(p_1) = 3\,S_1$ and $f^*(p_2) = 4\,S_2$, where $S_j = (f^{-1}(p_j))_{\mathrm{red}}$ is a smooth bielliptic surface, and the normal bundle $\mathcal{O}_X(S_j)|_{S_j}$ has order exactly m_j in $\operatorname{Pic}(S_j)$.

(4) X is simply connected and $H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z})$; in particular $b_2(X) = b_3(X) = 0$ and $e(X) = c_3(X) = 2 = e(W)$. Consequently X is diffeomorphic to the 6-sphere.

(5) $a(X) = 1$, and f is the algebraic reduction of X: $\mathcal{M}(X) = f^*\mathcal{M}(B) = f^*\mathbb{C}(t)$, that is, every meromorphic function on X is the pull back under f of a rational function on B; moreover the very general fibre of f has algebraic dimension 0.

(6) $f_*\mathcal{O}_X = \mathcal{O}_B$, $R^1 f_*\mathcal{O}_X \cong \mathcal{O}_B \oplus \mathcal{O}_B(-1)$ and $R^2 f_*\mathcal{O}_X \cong \mathcal{O}_B(-1)$; moreover
$$h^{0,1}(X) = 1, \qquad h^{0,2}(X) = h^{0,3}(X) = 0, \qquad h^{1,0}(X) = h^{2,0}(X) = h^{3,0}(X) = 0.$$
In particular $\chi(\mathcal{O}_X) = 0$; $\operatorname{Pic}(X) = \operatorname{Pic}^0(X) \cong H^1(X, \mathcal{O}_X) \cong \mathbb{C}$, since $H^1(X;\mathbb{Z}) = H^2(X;\mathbb{Z}) = 0$; and the Frölicher spectral sequence of X does not degenerate at E_1, since $b_1(X) = 0 < h^{0,1}(X)$.

(7) $K_X \cong f^*\mathcal{O}_B(-1) \otimes \mathcal{O}_X(2S_2)$; in particular K_X is not a torsion element of $\operatorname{Pic}(X)$, because $\mathcal{O}_X(4S_2) = f^*\mathcal{O}_B(1)$ gives $K_X^{\otimes 4k} \cong f^*\mathcal{O}_B(-2k)$, whose space of sections is $H^0(B, \mathcal{O}_B(-2k)) = 0$ for $k \geq 1$ by $f_*\mathcal{O}_X = \mathcal{O}_B$, while $h^0(\mathcal{O}_X) = 1$ shows that the trivial bundle does have sections. The Chern classes satisfy $c_1(X) = c_2(X) = 0$ in $H^*(X;\mathbb{Z})$, and the Chern numbers are $c_1 c_2 = 0$ and $c_3 = 2$, whence $\chi(X, T_X) = 1$ by Hirzebruch–Riemann–Roch.

(8) $h^0(X, T_X) = 1$ and $\operatorname{Aut}^0(X) \cong \mathbb{C}^*$, acting along the fibres of f; the generating vector field is the vertical field ξ determined by the monodromy invariant vector $\hat{\delta}$ of the period lattice, and the fixed locus of the action is a single double curve of W, a smooth rational curve (cf. [HKP, Prop. 2.5]).

Corollary 1.1. The six-sphere S^6 carries an integrable complex structure, namely the one obtained by transporting the complex structure of X along a diffeomorphism $X \to S^6$.

Remark 1.2. Since $b_2(X) = 0$, X is not Kähler, is not Moishezon [CDP98, Lemma 1.4], and does not belong to Fujiki's class $\mathcal{C}$. The invariants in items (6)–(8) agree with the known constraints on a compact complex threefold homeomorphic to S^6: $h^0(T_X) = 1 \leq 2$ [HKP, Cor. 1.3]; $h^{0,1} = h^{0,2} + 1$, $H^3(X, \mathcal{O}_X) = 0$ and $\operatorname{Pic}(X) = \operatorname{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)$ [HKP, Cor. 10.2]; K_X not torsion, as forced in [HKP, §10]; and the fixed locus of the $\mathbb{C}^*$-action of item (8), computed in §9, is of one of the two shapes permitted by [HKP, Prop. 2.5].

Remark 1.3 (Hodge numbers not computed here). All the Hodge numbers of X are determined by item (6) except $h^{1,1}$ and $h^{1,2} = h^{2,1}$. Since $\sum_{p,q}(-1)^{p+q}h^{p,q} = e(X)$ on any compact complex manifold (the Euler characteristic is unchanged through the Frölicher spectral sequence), equivalently by the Hirzebruch–Riemann–Roch identity $\sum_p(-1)^p\chi(\Omega^p_X) = c_3(X)$ (§9.4), the values $\chi(\mathcal{O}_X) = 0$, $e(X) = 2$ and the vanishings of item (6) force $h^{1,1} = h^{2,2} = h^{1,2} + 1$. Only $h^{1,2}$ is left undetermined; we expect $h^{1,2} = 1$ (whence $h^{1,1} = h^{2,2} = 2$). A computation would require the direct images $R^q f_*\Omega^1_X$ and $R^q f_*\Omega^2_X$, hence a description of the sheaves $\Omega^p_{X/B}$ along the three singular fibres, where they have torsion, together with an analysis of the corresponding Leray spectral sequences in the light of the non-degeneration recorded in item (6).

The construction is explicit. Over B° one has a family $J \to B^\circ$ of 2-tori whose period matrix is written down in closed form in terms of three functions τ, μ, β on the upper half plane $\mathfrak{h}_z$ uniformising the orbifold $(B; 3, 4, \infty)$: τ is the elliptic modular parameter, $j(\tau) = 1728\,t$, while μ and β solve prescribed inhomogeneous transformation laws. The local monodromies at p_1 and p_2 have orders 3 and 4, and the local monodromy at p_0 is unipotent with square-zero logarithm. The three fibres over Σ are filled in by three local models: at p_0 by a quotient of an infinite smooth toric threefold built on the A_2-triangulation of the plane, in the spirit of Mumford's construction of degenerating abelian varieties [Mum72, AMRT, KKMS]; at p_1 and p_2 by logarithmic transforms in the sense of Kodaira [Kod64], with twist vectors $v_1 = \varepsilon$ and $v_2 = -\varepsilon'$, explicit vectors of the period lattice fixed by the respective local monodromies. The three twists are what makes X simply connected: §7 gives $\pi_1(X) \cong \mathbb{Z}/|p|\,\mathbb{Z}$ with $p = 12\ell_0 - 4\ell_1 - 3\ell_2$, where (ℓ_0, ℓ_1, ℓ_2) records the twist data, and the above choice yields $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$, so $p = -1$.

1.2 Relation to [CDP20]

The Main Theorem is incompatible with a published result. Campana, Demailly and Peternell proved in [CDP98] that a compact complex threefold X with $b_2(X) = 0$ and $a(X) > 0$ has $c_3(X) = e(X) = 0$, and deduced that a compact complex threefold homeomorphic to S^6 has algebraic dimension 0. That proof rests on [CDP98, Lemma 1.5], which is incorrect as stated; the corrigendum [CDP20] says so and re-proves the results in two cases. [CDP20, Thm. 2.1] treats the case in which there exists a non-constant meromorphic map $g\colon X \dashrightarrow \mathbb{P}^1$ which is not holomorphic (this case is written out in detail in [LRS, Thm. 3.1, Cor. 3.2], following [CDP98]), while [CDP20, Thm. 2.2] treats the case $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$, $a(X) = 1$ with *holomorphic* algebraic reduction $f\colon X \to C$, concluding (a) $H^i(X, T_X \otimes M) = 0$ for $i \neq 1$ and generic $M \in \mathrm{Pic}^0(X)$, (b) $\chi(X, T_X \otimes M) \leq 0$ and (c) $c_3(X) \leq 0$; together these give [CDP20, Cor. 2.3]: if X is homeomorphic to S^6 then $a(X) = 0$. The manifold of the Main Theorem has $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$, $a(X) = 1$, holomorphic algebraic reduction $f\colon X \to \mathbb{P}^1$, and $c_3(X) = 2 > 0$, so that conclusion (c) fails; so does conclusion (b), since $c_1(X) = c_2(X) = 0$ and $\mathrm{Pic}(X) = \mathrm{Pic}^0(X)$ give, by Riemann–Roch, $\chi(X, T_X \otimes M) = \frac{1}{2}c_3(X) = 1 > 0$ for *every* $M \in \mathrm{Pic}(X)$. It therefore contradicts [CDP20, Thm. 2.2] and [CDP20, Cor. 2.3], as well as [CDP20, Lemma 4.2] (the special case $g = 0$, $b_1 = 0$ of [CDP98, Lemma 3.2]), [CDP20, Cor. 5.2] and [CDP20, Prop. 5.5] (= [CDP98, Thm. 3.1, Rem. 3.8(1)]), whose conclusions fail for X (their printed proofs presuppose trivial monodromy); this failure is repairable for the purpose of the proof of Cor. 2.3 at X (§10), unlike the reduction at the non-normal fibre, which is decisive. It does not contradict [CDP20, Thm. 2.1] or [LRS], whose hypothesis is the existence of *some* non-constant non-holomorphic meromorphic map $X \dashrightarrow \mathbb{P}^1$, and no such map exists on our X: by item (5) of the Main Theorem $\mathcal{M}(X) = f^*\mathbb{C}(t)$, so every non-constant meromorphic map $X \dashrightarrow \mathbb{P}^1$ is of the form $h \circ f$ with h a non-constant rational function on B, hence is holomorphic.

Section 10 is devoted to this conflict and locates the step at which the argument of [CDP20] and the example part company. The key point is the fibre $W = f^{-1}(p_0)$: it is a normal crossings divisor which is not normal, the six sides of its hexagon being glued to each other in three opposite pairs (item (2) of the Main Theorem). We prove (Theorem 10.5) that for *every* holomorphic line bundle L on X one has $R^2 f_*(T_X \otimes L) \neq 0$, the obstruction being supported at p_0 and arising from the section $df \otimes e$ of a sheaf supported on W which exists precisely because W is non-normal. The vanishing of this second direct image is the hypothesis under which [CDP20, Prop. 2.4] is invoked in the proof of [CDP20, Thm. 2.2]; for X it fails, for every L, and so the proof of [CDP20, Thm. 2.2] does not cover X. Section 10 also shows, in the light of the monodromy group of f computed in §3, that the conclusions of [CDP20, Lemma 4.2], [CDP20, Cor. 5.2] and [CDP20, Prop. 5.5] fail for X as well. Section 10 discusses the two steps of the printed proofs that do not apply to X; the two statements cannot both be correct. For this reason the construction is given with every matrix, chart and identification explicit.

1.3 Plan of the paper

Section 2 fixes the linear algebra: the lattice $V \cong \mathbb{Z}^4$ with basis (γ, u, w, δ), the matrices T_1, T_2 of orders 3 and 4, the unipotent $T_0 = (T_1T_2)^{-1} = I + N$ with $N^2 = 0$, and the dual picture on $\Lambda = V^*$ with $A_j = \mathfrak{A}(T_j)$ and M_0. All facts used later are proved there: $\Lambda_{\mathrm{tor}} = \ker(M_0 - I) = \mathrm{im}(M_0 - I)$, the unimodularity of $B_0 \colon \overline{\Lambda} \to \Lambda_{\mathrm{tor}}$, the descriptions $\Lambda^{A_1} = \langle \varepsilon, \hat{\delta} \rangle$, $\Lambda^{A_2} = \langle \varepsilon', \hat{\delta} \rangle$, and the presentation $\Delta = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1 \rangle \cong \mathbb{Z}/3 * \mathbb{Z}/4$ of the orbifold fundamental group of $(B; 3, 4, \infty)$, the $(3, 4, \infty)$ triangle group, together with the representation $\rho_V \colon \Delta \to GL(V)$, $\rho_V(g_j) = T_j$. The group Δ is not isomorphic to $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$; there is a surjection $\Delta \to PSL_2(\mathbb{Z})$, and the modular parameter τ is equivariant along it.

Section 3 constructs the period family. Theorem 3.4 establishes existence and uniqueness, up to one additive constant c_0 which is then chosen, of τ, μ, β on $\mathfrak{h}_z$ subject to the equivariance and boundedness conditions listed there, their behaviour at the two elliptic points and at the cusp, the equivariance $\Pi(gz) = R_g(z)\Pi(z)M_g$ of the period matrix $\Pi = [Z \mid I]$, and the nondegeneracy $\mathrm{Im}\,\tau \cdot D \neq 0$ which makes $L(z) = \Pi(z)\Lambda$ a lattice in $\mathbb{C}^2$ for every z.[2] This yields the family $\mathcal{T} \to \mathfrak{h}_z$ of complex 2-tori, the quotient $J \to B^\circ$, its marking by Λ, and its monodromy.

Section 4 constructs the filling at p_0. The fan $\mathfrak{F}$ of cones over the A_2-triangulation of $\mathbb{R}^2 \times \{1\}$ gives a smooth toric threefold $Y_{\mathfrak{F}}$, not of finite type, with a global holomorphic function t_c; the group $\overline{\Lambda} \cong \mathbb{Z}^2$ acts by the maps $\Psi_{\bar{\lambda}} = (c_{\bar{\lambda}}(t_c), 1) \cdot \Phi_{\bar{\lambda}}$. Theorem 4.5 shows that for small radius this action is free and properly discontinuous, that N_0 is a complex threefold with f_0 proper, that the fibres over $t_c \neq 0$ are the tori of J under the tautological identification E_0, and that $W = f_0^{-1}(0)$ is as described in item (2). For the quotient to be a complex manifold proper over the disc, only $\det B_0 \neq 0$ and the holomorphy of $C(t_c)$ at $t_c = 0$ are used; it is the unimodularity $|\det B_0| = 1$ that makes the central fibre irreducible with normalisation a single dP_6. No positivity or symmetry of B_0 enters. Toric generalities are taken from [Ful93, Oda88, KKMS].

Section 5 constructs the fillings at p_1, p_2 by logarithmic transforms. Theorem 5.4 shows that $\tilde{g}_j^{\log}$, which in flat coordinates reads $(z, x) \mapsto (g_j z, A_j x + v_j/m_j)$, generates a free $\mathbb{Z}/m_j$-action on $\mathcal{T}|_{\Delta_j}$ — freeness holds because $3 \nmid \gamma(v_1)$ and $\gamma(v_2)$ is odd — that the quotient N_j is a complex threefold with $f_j^*(p_j) = m_j S_j$ and S_j bielliptic, and that translation by the logarithmic section σ_j conjugates $\tilde{g}_j^{\log}$ to $\tilde{g}_j^0$, identifying $N_j \setminus f_j^{-1}(p_j)$ with $J|_{D_j^*}$.

Section 6 glues J, N_0, N_1, N_2 and proves Theorem 6.2: X is a compact connected complex threefold, f is holomorphic and surjective, and X is independent, up to biholomorphism over B, of the auxiliary choices (radii, discs, branches of logarithm).

Section 7 computes the topology. Using the 0-section of J to split $\pi_1(J|_{B^\circ}) \cong \Lambda \rtimes \pi_1(B^\circ)$, van Kampen gives $\pi_1(X) \cong \mathbb{Z}/|p|\mathbb{Z}$ with $p = 12\ell_0 - 4\ell_1 - 3\ell_2$, hence $\pi_1(X) = 1$ here (Theorem 7.17); Mayer–Vietoris, with the homology of W and of the S_j computed from the cell structures of the appendix, gives $H_*(X; \mathbb{Z}) \cong H_*(S^6; \mathbb{Z})$ and $e(X) = 2$ (Theorem 7.22). The group $H_*(X; \mathbb{Z})$ is in fact computed twice in that section: once along the Mayer–Vietoris route just described, which uses the retraction and collapse of §7.2, and once along the Leray route of §7.7 together with Appendix B, which, in combination with Theorem 7.17 and Corollary 4.8, makes no use of §7.2; the recognition of X as S^6 therefore does not depend on Proposition 7.2. For comparison, the threefold X' obtained with $v_2 = +\varepsilon'$ has $\pi_1(X') \cong \mathbb{Z}/7\mathbb{Z}$.

Section 8 concludes that X is diffeomorphic to S^6 (Theorem 8.1): X is a closed smooth simply connected 6-manifold with the integral homology of S^6, hence a homotopy 6-sphere by Hurewicz and Whitehead [Hat02], hence homeomorphic to S^6 by [Sma61], and diffeomorphic to S^6 since $\Theta_6 = 0$ [KM63], with

[2] The manifold X depends on the choice of c_0; see Remark 6.4.

[Mil65].

Section 9 computes the invariants of Theorem 9.1. The algebraic dimension is determined by a direct computation on the fibres: from the explicit period matrix one finds that a class $E \in \Lambda^2 V$ lies in $NS(F_z)$ if and only if an explicit linear form $P_E(z)$ in the five functions $1, \tau, \mu, \beta, 6\mu^2 - \tau\beta$ vanishes, and these five functions are linearly independent over $\mathbb{C}$; hence for z outside a countable set one gets $NS(F_z) = \mathbb{Z}\eta$ with $\eta = u \wedge w + 6\,\gamma \wedge \delta$, whose hermitian form H_η has signature $(1,1)$, so that neither η nor $-\eta$ is the class of a positive line bundle, while $\eta^2 = 12 \cdot \mathrm{vol} \neq 0$ excludes $a(F_z) = 1$ as well; the very general fibre therefore has algebraic dimension 0. Every meromorphic function on X is therefore constant on the very general fibre, whence $\mathcal{M}(X) = f^*\mathcal{M}(B) = f^*\mathbb{C}(t)$ and $a(X) = 1$, with f the algebraic reduction. Section 9 also gives a second proof, by exclusion: $a(X) \geq 1$ because f^* embeds $\mathcal{M}(B)$ into $\mathcal{M}(X)$; $a(X) \neq 3$ because a Moishezon threefold has $b_2 > 0$ [CDP98, Lemma 1.4] while $b_2(X) = 0$ by §7; and $a(X) \neq 2$, since otherwise X carries a meromorphic map to a curve which is not holomorphic (as in the proof of [CDP20, Cor. 2.3]), whence $c_3(X) = 0$ by [CDP20, Thm. 2.1] (see also [LRS, Thm. 3.1]), contradicting $c_3(X) = e(X) = 2$. This route also yields the identification $\mathcal{M}(X) = f^*\mathcal{M}(B)$, as explained in Remark 9.9: once $a(X) = 1$, the field $\mathcal{M}(X)$ is algebraic over the subfield $f^*\mathcal{M}(B)$ of transcendence degree 1, and $f^*\mathcal{M}(B)$ is algebraically closed in $\mathcal{M}(X)$ because f has connected fibres [Ue75, §3]; hence $\mathcal{M}(X) = f^*\mathcal{M}(B)$ and f is the algebraic reduction. The section then computes the direct images $R^q f_*\mathcal{O}_X$ and the numbers $h^{0,q}$ of item (6), the vanishing of the holomorphic forms $h^{1,0} = h^{2,0} = h^{3,0} = 0$, the canonical bundle of item (7), the Chern numbers, and the vertical $\mathbb{C}^*$-action with $h^0(T_X) = 1$ and its fixed locus. Section 10 is the discussion of §1.2; the appendix collects the matrices, the fan $\mathfrak{F}$, and the cell structures used in §7.

1.4 Context

The question whether S^6 admits a complex structure goes back to Hopf [Hopf48]; for its history and the state of the art we refer to the surveys [AGr, Ang18]. Among the even-dimensional spheres only S^2 and S^6 carry almost complex structures; the structure on S^6 induced by octonionic multiplication is not integrable, its Nijenhuis tensor being nowhere zero, and the space of almost complex structures inducing a given orientation is connected, so that no homotopy-theoretic invariant can separate a hypothetical integrable structure from the octonionic one. The one general structural theorem in this direction is that of LeBrun [LeB87], anticipated by Blanchard: no complex structure on S^6 is orthogonal with respect to the round metric; the proof shows that such a structure would make S^6 Kähler, contradicting $b_2 = 0$, and the later refinements also use only $H^2(S^6) = 0$. The structure constructed here is not produced from a metric and carries no a priori relation to the round one, so [LeB87] does not bear on it.

Compact complex threefolds with $b_2 = 0$ are classical: Calabi and Eckmann [CE53] put complex structures on $S^3 \times S^3$ and on products of odd-dimensional spheres; these, the Hopf threefolds $S^1 \times S^5$, the quotients $SL(2,\mathbb{C})/\Gamma$ and the principal elliptic bundles of [LRS, §4] furnish many examples, with $a = 0, 1, 2$ (see [CDP98, §4]). What distinguishes S^6 among these is $e = 2 \neq 0$. The complex-analytic constraints on a hypothetical complex structure on S^6 have been studied in detail: Huckleberry, Kebekus and Peternell [HKP] proved that such an X is not almost homogeneous, that $h^0(T_X) \leq 2$, $H^3(X, \mathcal{O}_X) = 0$, $h^{0,1} = h^{0,2} + 1$ and $\mathrm{Pic}(X) = \mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)$, and analysed the possible fixed-point sets of a $\mathbb{C}^*$-action; Lehn, Rollenske and Schinko [LRS] obtained bounds on spaces of sections, and wrote out in detail the meromorphic-map case of [CDP98, CDP20], following [CDP98]. Constructions of complex structures on S^6 have been claimed by Etesi [Ete15] by entirely different means, from Samelson structures on G_2 and conjugate orbits; nothing here depends on them or bears on their validity.

The two mechanisms combined here are classical. Logarithmic transformations were introduced by Kodaira for elliptic surfaces [Kod64] (see also [BHPV, V.13]), where they are the standard device for changing the fundamental group of an elliptic fibration without changing the base; the Dolgachev surface obtained from a rational elliptic surface by logarithmic transformations of coprime multiplicities is simply connected [Dol81], and the present construction is an analogue one dimension up, with 2-tori in place of elliptic curves. Filling in a degenerating family of abelian varieties by a toroidal model over the nilpotent monodromy cone goes back to Mumford [Mum72], and in the arithmetic setting to [AMRT]; the combinatorial input used here is the A_2-triangulation of the plane. What is specific to the present situation is that the two mechanisms are used simultaneously, over a base $\mathbb{P}^1$ with exactly three critical points, for a representation ρ_V of the $(3,4,\infty)$ triangle group whose local monodromies have orders 3, 4 and ∞, and that the resulting central fibre W at the unipotent point is non-normal, with $e(W)=2$ accounting for the whole Euler characteristic of X.

2 Lattice and monodromy data.

2.1 The lattices V and Λ

As on page 1 (Setup), V is free of rank 4 with ordered basis (γ,u,w,δ) and $\Lambda := V^*$ carries the dual basis $(\hat\gamma,\hat u,\hat w,\hat\delta)$; we write $\langle\lambda,x\rangle := \lambda(x)$, elements are columns, matrices act on columns from the left, and the columns of a matrix are the images of the basis vectors. Evaluation at γ is the functional

$$\gamma\colon \Lambda\to\mathbb{Z},\qquad \gamma(a\hat\gamma+b\hat u+c\hat w+d\hat\delta)=a,\qquad \ker\gamma=\langle\hat u,\hat w,\hat\delta\rangle.$$

For $T\in GL(V)$ put

$$\mathfrak{A}(T) := (T^{-1})^t \in GL(\Lambda) \tag{2.1}$$

(matrix in the dual basis): the unique element of $GL(\Lambda)$ with $\langle\mathfrak{A}(T)\lambda, Tx\rangle = \langle\lambda,x\rangle$ for all λ,x, i.e. the contragredient $(T^t)^{-1}$ of T.

2.2 The monodromy matrices

Definition 2.1. In the basis (γ,u,w,δ) set

$$T_1 := \begin{pmatrix}1&0&-6&2\\0&-1&1&1\\0&-1&0&1\\0&0&0&1\end{pmatrix},\qquad T_2 := \begin{pmatrix}1&6&0&-3\\0&0&-1&1\\0&1&0&0\\0&0&0&1\end{pmatrix},$$

that is,

$$\begin{aligned}&T_1\gamma=\gamma,\quad T_1u=-u-w,\quad T_1w=-6\gamma+u,\quad T_1\delta=2\gamma+u+w+\delta,\\ &T_2\gamma=\gamma,\quad T_2u=6\gamma+w,\quad T_2w=-u,\qquad\quad T_2\delta=-3\gamma+u+\delta.\end{aligned} \tag{2.2}$$

Let $G := \langle T_1,T_2\rangle \subset GL(V)$.

Lemma 2.2. (i) $\det T_1=\det T_2=1$. (ii) $T_1^3=I\neq T_1$ and $T_2^4=I\neq T_2^2$, so T_1,T_2 have exact orders 3, 4. (iii) Put $T_0 := (T_1T_2)^{-1}$ and $N := T_0 - I$. Then $T_0 = I+N$, $N^2=0$, $T_1T_2T_0=I$, and

$$N\gamma=Nu=0,\quad Nw=-u,\quad N\delta=\gamma,\qquad \ker N=\operatorname{im}N=\langle\gamma,u\rangle;$$

explicitly

$$T_0=\begin{pmatrix}1&0&0&1\\0&1&-1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Proof. Direct computation from (2.2):

$$\begin{aligned}T_1^2&\colon\ \gamma\mapsto\gamma,\ u\mapsto 6\gamma+w,\ w\mapsto -6\gamma-u-w,\ \delta\mapsto -2\gamma+u+\delta,\\ T_2^2&\colon\ \gamma\mapsto\gamma,\ u\mapsto 6\gamma-u,\ w\mapsto -6\gamma-w,\ \delta\mapsto u+w+\delta,\\ T_1T_2&\colon\ \gamma\mapsto\gamma,\ u\mapsto u,\ w\mapsto u+w,\ \delta\mapsto -\gamma+\delta,\end{aligned}$$

whence $T_1^2\neq I\neq T_2^2$, $T_1^3=I$, $T_2^4=I$, and $T_1T_2=I-N$ with N as stated. Both T_j fix γ, preserve $\langle\gamma,u,w\rangle$ and act trivially on $V/\langle\gamma,u,w\rangle$, and on $\langle\gamma,u,w\rangle/\langle\gamma\rangle$ by $\bar u\mapsto -\bar u-\bar w$, $\bar w\mapsto\bar u$ resp. $\bar u\mapsto\bar w$, $\bar w\mapsto-\bar u$, of determinant 1; this gives (i). Since N maps w,δ into $\ker N=\langle\gamma,u\rangle=\operatorname{im}N$ we get $N^2=0$, hence $(I-N)(I+N)=I$ and $T_0=I+N$. □

2.3 The dual matrices and the distinguished sublattices of Λ

Definition 2.3. Put $A_j := \mathfrak{A}(T_j)$ $(j = 1, 2)$ and $M_0 := \mathfrak{A}(T_0)$.

Lemma 2.4. $\mathfrak{A}\colon GL(V) \to GL(\Lambda)$ is a group homomorphism, and in the dual basis

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}, \quad M_0 = I - N^t,$$

that is,

$$\begin{aligned} &A_1\hat\gamma = \hat\gamma + 6\hat u - 6\hat w - 2\hat\delta, && A_1\hat u = -\hat w + \hat\delta, && A_1\hat w = \hat u - \hat w, && A_1\hat\delta = \hat\delta, \\ &A_2\hat\gamma = \hat\gamma - 6\hat w + 3\hat\delta, && A_2\hat u = \hat w, && A_2\hat w = -\hat u + \hat\delta, && A_2\hat\delta = \hat\delta, \\ &M_0\hat\gamma = \hat\gamma - \hat\delta, && M_0\hat u = \hat u + \hat w, && M_0\hat w = \hat w, && M_0\hat\delta = \hat\delta. \end{aligned} \tag{2.3}$$

Moreover $A_1A_2M_0 = I$ and $\gamma \circ A_j = \gamma \circ M_0 = \gamma$.

Proof. $\mathfrak{A}(ST) = ((ST)^{-1})^t = \mathfrak{A}(S)\mathfrak{A}(T)$. By Lemma 2.2(ii), $A_1 = (T_1^2)^t$ and $A_2 = (T_2^3)^t$, and $T_2^3\colon \gamma \mapsto \gamma,\ u \mapsto -w,\ w \mapsto -6\gamma + u,\ \delta \mapsto 3\gamma + w + \delta$; transposing gives the displayed A_1, A_2. From $T_0^{-1} = I - N$ we get $M_0 = I - N^t$, and $N^t\hat\gamma = \hat\delta$, $N^t\hat u = -\hat w$, $N^t\hat w = N^t\hat\delta = 0$ by Lemma 2.2(iii), which is (2.3). Then $A_1A_2M_0 = \mathfrak{A}(T_1T_2T_0) = I$, and $\gamma(A_j\lambda) = \langle \mathfrak{A}(T_j)\lambda, T_j\gamma\rangle = \gamma(\lambda)$ since $T_j\gamma = \gamma$; likewise for M_0. □

Definition 2.5. Put $\Lambda_{\mathrm{tor}} := \langle \hat w, \hat\delta\rangle \subset \Lambda$ and $\overline{\Lambda} := \Lambda/\Lambda_{\mathrm{tor}}$ with basis $(\bar{\hat\gamma}, \bar{\hat u})$; identify $\overline{\Lambda} \cong \mathbb{Z}^2$ by $a\bar{\hat\gamma} + b\bar{\hat u} \mapsto (a, b)^t$ and $\Lambda_{\mathrm{tor}} \cong \mathbb{Z}^2$ by $c\hat w + d\hat\delta \mapsto (c, d)^t$. Set $\varepsilon := \hat\gamma + 2\hat u - 4\hat w$ and $\varepsilon' := \hat\gamma + 3\hat u - 3\hat w$.

Lemma 2.6. $\Lambda_{\mathrm{tor}} = \ker(M_0 - I) = \mathrm{im}(M_0 - I)$, and $M_0 - I$ induces

$$B_0\colon \overline{\Lambda} \to \Lambda_{\mathrm{tor}}, \qquad B_0\bar{\hat\gamma} = -\hat\delta, \quad B_0\bar{\hat u} = \hat w,$$

with matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the above identifications; in particular $\det B_0 = 1$, so B_0 is unimodular. Moreover $\Lambda^{A_1} := \ker(A_1 - I) = \langle \varepsilon, \hat\delta\rangle$ and $\Lambda^{A_2} := \ker(A_2 - I) = \langle \varepsilon', \hat\delta\rangle$ are saturated, $\gamma(\varepsilon) = \gamma(\varepsilon') = 1$, and with the twist vectors

$$v_1 := \varepsilon \in \Lambda^{A_1}, \qquad v_2 := -\varepsilon' \in \Lambda^{A_2}$$

one has $(\ell_0, \ell_1, \ell_2) := (0, \gamma(v_1), \gamma(v_2)) = (0, 1, -1)$.

Proof. By (2.3), for $\lambda = a\hat\gamma + b\hat u + c\hat w + d\hat\delta$,

$$(M_0 - I)\lambda = -N^t\lambda = -a\hat\delta + b\hat w.$$

Hence $(M_0 - I)\lambda = 0$ if and only if $a = b = 0$, i.e. $\lambda \in \Lambda_{\mathrm{tor}}$; and the image is exactly $\langle \hat w, \hat\delta\rangle = \Lambda_{\mathrm{tor}}$, the pair (a, b) ranging over all of $\mathbb{Z}^2$. Thus $M_0 - I$ kills Λ_{tor} and the induced map B_0 sends $\bar{\hat\gamma} \mapsto -\hat\delta$, $\bar{\hat u} \mapsto \hat w$, i.e. $(a, b)^t \mapsto (b, -a)^t$ in the chosen coordinates, of determinant 1.

For λ as above, (2.3) gives

$$\begin{aligned} (A_1 - I)\lambda &= a(6\hat u - 6\hat w - 2\hat\delta) + b(-\hat u - \hat w + \hat\delta) + c(\hat u - 2\hat w), \\ (A_2 - I)\lambda &= a(-6\hat w + 3\hat\delta) + b(-\hat u + \hat w) + c(-\hat u - \hat w + \hat\delta), \end{aligned}$$

so $(A_1 - I)\lambda = 0$ reads $6a - b + c = -6a - b - 2c = -2a + b = 0$, i.e. $b = 2a$, $c = -4a$, d free, and $(A_2 - I)\lambda = 0$ reads $b + c = -6a + b - c = 3a + c = 0$, i.e. $b = 3a$, $c = -3a$, d free. These kernels are saturated, being kernels of endomorphisms of the torsion-free group Λ, and $\gamma(\varepsilon) = \gamma(\varepsilon') = 1$ by definition of γ. □

Lemma 2.7. Let G act on V tautologically and on Λ through $\mathfrak{A}$. Then:

(i) $V^{T_1} = \langle \gamma,\ 2u + w + 3\delta\rangle$, $V^{T_2} = \langle \gamma,\ u + w + 2\delta\rangle$, $V^G = \mathbb{Z}\gamma$;

(ii) $\Lambda^G = \Lambda^{A_1} \cap \Lambda^{A_2} = \mathbb{Z}\hat\delta$;

(iii) every $g \in G$ preserves $\langle \gamma, u, w\rangle$ and acts trivially on $V/\langle \gamma, u, w\rangle$, and every $\mathfrak{A}(g)$ satisfies $\gamma \circ \mathfrak{A}(g) = \gamma$; with $\Sigma_V := \sum_{g\in G} \operatorname{im}(g - I)$ and $\Sigma_\Lambda := \sum_{g\in G} \operatorname{im}(\mathfrak{A}(g) - I)$,
$$\Sigma_V = \langle \gamma, u, w\rangle, \qquad \Sigma_\Lambda = \ker \gamma,$$
so the coinvariants $V_G \cong \mathbb{Z}$ (generated by the class of δ) and $\Lambda_G \cong \mathbb{Z}$ (via γ) are free of rank 1; equivalently $[(T_1 - I) \mid (T_2 - I)]$ and $[(A_1 - I) \mid (A_2 - I)]$ have Smith normal form $\operatorname{diag}(1, 1, 1, 0)$;

(iv) the smallest $\langle A_1, A_2\rangle$-invariant subgroup of Λ containing Λ_{tor} is $\ker \gamma$.

Proof. (i) For $x = a\gamma + bu + cw + d\delta$, (2.2) gives
$$T_1 x = (a - 6c + 2d)\gamma + (-b + c + d)u + (-b + d)w + d\delta,$$
$$T_2 x = (a + 6b - 3d)\gamma + (-c + d)u + bw + d\delta,$$
so $T_1 x = x$ iff $d = 3c$, $b = 2c$, and $T_2 x = x$ iff $d = 2b$, $c = b$. Comparing $a\gamma + c(2u + w + 3\delta) = a'\gamma + b(u + w + 2\delta)$ forces $b = c = 0$, so $V^G = \mathbb{Z}\gamma$.

(ii) By Lemma 2.6 an element of the intersection is $a\varepsilon + d\hat{\delta} = a'\varepsilon' + d'\hat{\delta}$; the $\hat{\gamma}$- and $\hat{u}$-coefficients give $a = a'$ and $2a = 3a'$, so $a = 0$.

(iii) By (2.2) each generator T_j preserves $U := \langle \gamma, u, w\rangle$ and acts trivially on V/U; these two properties are stable under composition and inversion, hence hold for every $g \in G$, so $\operatorname{im}(g - I) \subseteq U$ and $\Sigma_V \subseteq U$. Likewise $\gamma \circ A_j = \gamma$ (Lemma 2.4) gives $\gamma \circ \mathfrak{A}(g) = \gamma$ for every $g \in G$, whence $\operatorname{im}(\mathfrak{A}(g) - I) \subseteq \ker \gamma$ and $\Sigma_\Lambda \subseteq \ker \gamma$. For the reverse inclusions it suffices to use the generators: Σ_V contains the images of the basis vectors under $T_j - I$, namely
$$-2u - w, \qquad -6\gamma + u - w, \qquad 2\gamma + u + w,$$
$$6\gamma - u + w, \qquad -u - w, \qquad -3\gamma + u,$$
and $(-2u - w) - (-u - w) = -u$, then w, then 2γ and 3γ, hence γ, lie in Σ_V; while $(A_1 - I)\hat{w} + (A_2 - I)\hat{u} = -\hat{w}$, then $\hat{u}$, then $\hat{\delta} = (A_1 - I)\hat{u} + \hat{u} + \hat{w}$ lie in Σ_Λ. Both Σ_V and Σ_Λ are spanned by parts of the bases, hence are direct summands with free rank-1 quotients.

(iv) Such a subgroup contains $\hat{w}$ and $A_1\hat{w} = \hat{u} - \hat{w}$, hence $\hat{u}$ and $\hat{\delta}$, i.e. $\ker \gamma$; and $\ker \gamma$ is invariant since $\gamma \circ A_j = \gamma$. $\square$

2.4 The invariant alternating form

Lemma 2.8. The group of G-invariant alternating forms on V is infinite cyclic, generated by Q_0 with
$$Q_0(\gamma, \delta) = 1, \quad Q_0(u, w) = 6, \quad Q_0(\gamma, u) = Q_0(\gamma, w) = Q_0(u, \delta) = Q_0(w, \delta) = 0;$$
thus Q_0 is, up to sign, the unique primitive invariant alternating form. Moreover $b(x, y) := Q_0(x, Ny)$ is symmetric and descends to $V/\ker N \cong \mathbb{Z}^2$; in the basis given by the classes of w, δ its Gram matrix is
$$\operatorname{diag}(6, -1),$$
so b is nondegenerate of signature $(1, 1)$ and discriminant -6.

Proof. Write $q_{xy} := Q(x, y)$ for an invariant alternating Q. By (2.2), T_1-invariance on the pairs (γ, u) and (γ, w) reads
$$-q_{\gamma u} - q_{\gamma w} = q_{\gamma u}, \qquad q_{\gamma u} = q_{\gamma w},$$
i.e. $2q_{\gamma u} + q_{\gamma w} = 0$ and $q_{\gamma u} = q_{\gamma w}$, whence $3q_{\gamma u} = 0$ and
$$q_{\gamma u} = q_{\gamma w} = 0.$$
Using these first two equations to drop all terms in $q_{\gamma u}, q_{\gamma w}$ from the remaining expansions, T_1-invariance on the pairs (u, δ) and (w, δ) reads
$$-q_{u\delta} - q_{w\delta} = q_{u\delta}, \qquad -6q_{\gamma\delta} + q_{uw} + q_{u\delta} = q_{w\delta},$$
and, in the same way (again using the first two equations), T_2-invariance on the same four pairs reads
$$q_{\gamma w} = q_{\gamma u}, \quad -q_{\gamma u} = q_{\gamma w}, \quad -q_{u\delta} = q_{w\delta}, \quad 6q_{\gamma\delta} - q_{uw} + q_{w\delta} = q_{u\delta};$$

the remaining pairs give no condition. Hence $q_{\gamma u} = q_{\gamma w} = q_{u\delta} = q_{w\delta} = 0$ and $q_{uw} = 6q_{\gamma\delta}$, i.e. $Q \in \mathbb{Z}Q_0$, and Q_0 is invariant. For b: $\ker N = \operatorname{im} N = \langle \gamma, u\rangle$ and Q_0 vanishes on $\langle \gamma, u\rangle \times \langle \gamma, u\rangle$, so $b(x,y) = 0$ whenever x or y lies in $\ker N$, and b descends; finally

$$b(w,w) = Q_0(w,-u) = 6, \qquad\qquad b(\delta,\delta) = Q_0(\delta,\gamma) = -1,$$
$$b(w,\delta) = Q_0(w,\gamma) = 0 = Q_0(\delta,-u) = b(\delta,w).$$

□

Remark 2.9. In §4 it is the unimodularity of B_0 that is used: the lattice $B_0\overline{\Lambda}$ of cocharacter translations is all of Λ_{tor}, and this is what makes the toric filling have an irreducible central fibre W containing a single copy of dP_6; for $|\det B_0| = k > 1$ the same construction would produce k such pieces. The form Q_0 is used again in Remark 3.23, in §7 through the invariant bivector $\eta = u \wedge w + 6\,\gamma \wedge \delta$ dual to Q_0, and in Appendix A. It is recorded here already in order to place the present data in context: for a one-parameter degeneration of *polarized* abelian varieties with unipotent monodromy $\exp N$ and polarization Q the form $x \mapsto Q(x, Nx)$ on $V/\ker N$ is definite ([Cle77], [Mum72]), whereas by Lemma 2.8 the pair (Q_0, N) has signature $(1,1)$.

2.5 The base orbifold, the group Δ, and the representation ρ_V

With the conventions of page 1 (Setup): $B = \mathbb{P}^1$ with coordinate t, $p_1\colon t = 0$, $p_2\colon t = 1$, $p_0\colon t = \infty$ with $t_c = 1/t$, $B^\circ = B \setminus \{p_0, p_1, p_2\}$, $m_1 = 3$, $m_2 = 4$, $\zeta_j = e^{-2\pi i/m_j}$. Let $\mathcal{O}$ be the orbifold with underlying space B, cone points of orders 3 at p_1 and 4 at p_2, and a puncture at p_0, i.e. of signature $(0; 3, 4, \infty)$.

Lemma 2.10. Let $g \in PSL_2(\mathbb{R})$ be elliptic of order $m < \infty$ with fixed point $z_0 \in \mathfrak{h}$, and $c(z) := (z - z_0)/(z - \bar{z}_0)$. Then c maps $\mathfrak{h}$ biholomorphically onto the unit disc with $c(z_0) = 0$, and $c \circ g \circ c^{-1}$ is the rotation $s \mapsto \zeta s$ with $\zeta = g'(z_0)$ a primitive m-th root of unity; in particular $c \circ g = \zeta \cdot c$.

Proof. c is a Möbius map carrying $\mathbb{R} \cup \{\infty\}$ to the unit circle and z_0 to 0; the disc automorphism $c \circ g \circ c^{-1}$ fixes 0, hence is $s \mapsto \zeta s$ with ζ its derivative at 0, equal to $g'(z_0)$, of order m. □

Proposition 2.11. There exist a copy $\mathfrak{h}_z$ of the upper half-plane, a Fuchsian group $\Delta \subset PSL_2(\mathbb{R})$ and a holomorphic surjection $\pi\colon \mathfrak{h}_z \to B \setminus \{p_0\}$ such that:

(i) Δ acts properly discontinuously, π induces $\mathfrak{h}_z/\Delta \cong B \setminus \{p_0\}$, π is a local biholomorphism off $\pi^{-1}\{p_1, p_2\}$ and has local degree m_j at each point of $\pi^{-1}(p_j)$; the stabiliser of such a point is cyclic of order m_j and all other stabilisers are trivial;

(ii) there are $z_j \in \pi^{-1}(p_j)$ and $g_1, g_2 \in \Delta$ with $g_j(z_j) = z_j$, g_j of order m_j and $g_j'(z_j) = \zeta_j$, such that
$$\Delta = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1\rangle \cong \mathbb{Z}/3 * \mathbb{Z}/4,$$
and such that $g_0 := (g_1 g_2)^{-1}$ is parabolic; by construction $g_1 g_2 g_0 = 1$;

(iii) if $* \in \partial\mathfrak{h}_z$ is the fixed point of g_0, its stabiliser in Δ is exactly $\langle g_0\rangle$, and for all small $r > 0$ the component U_r of $\pi^{-1}(\{0 < |t_c| < r\})$ whose closure contains $*$ is a $\langle g_0\rangle$-invariant, connected and simply connected cusp neighbourhood of $*$ squeezed between two horodiscs at $*$, i.e. $H'' \subseteq U_r \subseteq H'$ for suitable horodiscs H'', H' at $*$; shrinking r so that U_r lies in a strictly smaller horodisc at $*$, the set U_r is closed in $\pi^{-1}(\{0 < |t_c| < r\})$ and hence is a connected component of it, every other component lying in a translate gH' with $g \in \Delta \setminus \langle g_0\rangle$; it satisfies $gU_r \cap U_r = \varnothing$ for $g \in \Delta \setminus \langle g_0\rangle$, and π induces a biholomorphism $U_r/\langle g_0\rangle \cong \{0 < |t_c| < r\}$.

Proof. Since $\frac{1}{3} + \frac{1}{4} < 1$, $\mathcal{O}$ is hyperbolic, hence uniformised by a Fuchsian group of signature $(0; 3, 4, \infty)$ ([Thu97], Ch. 13; [Bea83], Ch. 10); explicitly, Δ is the orientation-preserving subgroup of the group generated by the reflections $\sigma_a, \sigma_b, \sigma_c$ in the sides $a = BC$, $b = CA$, $c = AB$ of the hyperbolic triangle T with vertices A, B, C counterclockwise and angles $\pi/3, \pi/4, 0$ (Poincaré's theorem; [Bea83], Ch. 7, 9–10). The quotient surface is a once-punctured sphere, i.e. $\mathbb{C}$; composing with the affine automorphism carrying the order-3 cone point to $0 = p_1$ and the order-4 one to $1 = p_2$ yields π, and (i) is the standard description of a Fuchsian group by its signature ([Bea83], Ch. 10).

(ii) Put $x := \sigma_b\sigma_c$, $y := \sigma_c\sigma_a$, $z := \sigma_a\sigma_b$, so $xyz = 1$ and $\Delta = \langle x, y, z \mid x^3 = y^4 = 1,\ xyz = 1\rangle$ ([Bea83], Ch. 10). The relevant rule is: if two geodesics α, β meet at P and θ denotes the angle from α to β measured counterclockwise, then the product $\sigma_\beta\sigma_\alpha$ (reflect in α first, then in β) is the counterclockwise rotation about P through 2θ ([Bea83], Ch. 7). Since the vertices A, B, C are counterclockwise, the counterclockwise angle at A from $c = AB$ to $b = CA$ is the interior angle $\pi/3$, and that at B from $a = BC$ to $c = AB$ is $\pi/4$; hence $x = \sigma_b\sigma_c$ is the counterclockwise rotation about A through $2\pi/3$, $y = \sigma_c\sigma_a$ the counterclockwise rotation about B through $2\pi/4$, and z is parabolic, fixing the ideal vertex C; by Lemma 2.10, $x'(A) = e^{2\pi i/3}$ and $y'(B) = e^{2\pi i/4}$. With

$$z_1 := A, \quad z_2 := B, \quad g_1 := x^{-1}, \quad g_2 := y^{-1}$$

the two rotations are reversed simultaneously, so that $g_1'(z_1) = \zeta_1$ and $g_2'(z_2) = \zeta_2$, of orders $3, 4$. Inverting $xyz = 1$ gives $g_2g_1 = z$, so $g_1g_2 = g_1zg_1^{-1}$ is parabolic, hence so is g_0; eliminating z gives the stated presentation.

(iii) *The stabiliser of $*$ is $\langle g_0\rangle$.* By the standard description of a cusp ([Bea83], Ch. 10) the stabiliser Δ_* of the fixed point of a parabolic element of a Fuchsian group is infinite cyclic, generated by a primitive parabolic h, and there is a horodisc H' at $*$ with $gH' \cap H' = \emptyset$ for $g \notin \Delta_*$. Write $g_0 = h^n$ with $n \geq 1$; we claim $n = 1$, i.e. that g_0, equivalently $g_0^{-1} = g_1g_2$, is not a proper power in Δ. Indeed, in the free product $\Delta = \langle g_1\rangle * \langle g_2\rangle$ the element g_1g_2 is cyclically reduced of length 2, and cyclically reduced length is a conjugacy invariant. An element $h \in \Delta$ of finite order is conjugate into a free factor, so all its powers have finite order and cannot be parabolic; hence h has infinite order and is conjugate to a cyclically reduced word h_0 of even length $\ell \geq 2$, so that h^n is conjugate to the cyclically reduced word h_0^n of length $n\ell$. If $n \geq 2$ this length is $\geq 4 > 2$, a contradiction. Thus $n = 1$ and $\Delta_* = \langle g_0\rangle$, and $H'/\langle g_0\rangle$ embeds in $\mathfrak{h}_z/\Delta$ as a punctured-disc neighbourhood of the puncture.

The cusp coordinate. Conjugate in $PSL_2(\mathbb{R})$ so that $* = \infty$ and $H' = \{\operatorname{Im}\zeta > C\}$; then $g_0(\zeta) = \zeta + \varepsilon_0$ with $\varepsilon_0 \in \mathbb{R}^\times$, and after the further conjugation $\zeta \mapsto \zeta/|\varepsilon_0|$ we may assume $\varepsilon_0 \in \{+1, -1\}$. The sign ε_0 is intrinsic, the elements of $PSL_2(\mathbb{R})$ fixing ∞ being the maps $\zeta \mapsto a\zeta + c$ with $a > 0$, whose conjugation action preserves it. Its value plays no role in the present proof, since $q := e^{2\pi i\zeta}$ is $\langle g_0\rangle$-invariant for either sign and identifies $H'/\langle g_0\rangle$ with a punctured disc D^*; the value is determined once the period map τ is taken into account, and Remark 2.12 gives $\varepsilon_0 = -1$. Thus π induces an injective holomorphic map $\psi\colon D \to B$ with $\psi(0) = p_0$, where we read B near p_0 in the coordinate t_c and set $\psi(0) = 0$. Choose $r_0 > 0$ with $\{|t_c| < r_0\} \subseteq \psi(D)$, and let $0 < r < r_0$. Then $V_r := \psi^{-1}(\{|t_c| < r\})$ is a simply connected neighbourhood of 0 in D, biholomorphic to a disc, and

$$U_r = \{\zeta \in H' \ :\ e^{2\pi i\zeta} \in V_r \smallsetminus \{0\}\}$$

is the component of $\pi^{-1}(\{0 < |t_c| < r\})$ accumulating at $*$: it is the preimage under $\zeta \mapsto e^{2\pi i\zeta}$ of a punctured simply connected domain around 0, hence connected, simply connected and $\langle g_0\rangle$-invariant, and π induces $U_r/\langle g_0\rangle \cong \{0 < |t_c| < r\}$; disjointness of its Δ-translates off $\langle g_0\rangle$ follows from that of $H' \supseteq U_r$. It is moreover the *only* component accumulating at $*$: since π is the quotient by Δ and U_r is a component of $\pi^{-1}(\{0 < |t_c| < r\})$ with trivial setwise stabiliser outside $\langle g_0\rangle$, every component of $\pi^{-1}(\{0 < |t_c| < r\})$ is a translate gU_r with $g \in \Delta$, and every component other than U_r is such a translate with $g \notin \langle g_0\rangle$, hence lies in gH'; since $U_r \subseteq H'$ is contained in a horodisc at $*$, the closure of gU_r meets $\partial\mathfrak{h}_z$ only at $g*$, and $g* \neq *$ because the stabiliser of $*$ is exactly $\langle g_0\rangle$. Hence no other component accumulates at $*$. Finally, choosing $0 < \varrho < \varrho'$ with $\{|q| < \varrho\} \subseteq V_r \subseteq \{|q| < \varrho'\}$ gives horodiscs $H'' = \{\operatorname{Im}\zeta > \log(1/\varrho)/2\pi\} \subseteq U_r$ and $U_r \subseteq \{\operatorname{Im}\zeta > \log(1/\varrho')/2\pi\} \cap H'$, as claimed. □

Remark 2.12 (The translation sign at the cusp). The normalisation $g_0(\zeta) = \zeta + \varepsilon_0$, $\varepsilon_0 \in \{\pm 1\}$, of the proof of Proposition 2.11(iii) concerns the *uniformising coordinate* ζ on $\mathfrak{h}_z$, whereas the relation $\tau \circ g_0 = \tau - 1$ of §3 concerns the *period map* $\tau\colon \mathfrak{h}_z \to \mathfrak{h}$ and expresses its equivariance along ρ_Λ, i.e. $\rho_\Lambda(g_0) = M_0$. The two maps are distinct, but the second relation determines the first sign: we show that $\varepsilon_0 = -1$.

Indeed, τ is equivariant along the surjection $\Delta \to PSL_2(\mathbb{Z})$ of Remark 2.13, so $j \circ \tau = 1728\, t \circ \pi$ on $\mathfrak{h}_z$; equivalently, on U_r the function $e^{2\pi i\tau}$ is, through the identifications of Proposition 2.11(iii), the local parameter of the cusp of $X(1)$ pulled back along the map ψ, which has a simple zero at 0. Writing $q := e^{2\pi i\zeta}$, this means that there are a holomorphic unit u_1 near 0 (the ratio of the two local cusp parameters, coming from the local biholomorphism j-chart versus t_c-chart at the cusp) and the expansion $\psi(v) = v\,w(v)$ with $w(0) \neq 0$, such that

$$e^{2\pi i\tau} = u_1\big(\psi(q)\big)\,\psi(q), \qquad \text{so} \qquad e^{2\pi i(\tau-\zeta)} = V(q), \quad V(v) := u_1\big(\psi(v)\big)\,w(v),$$

where V is holomorphic and nowhere zero on a neighbourhood of 0 in V_r; in particular $e^{2\pi i(\tau-\zeta)}$ extends across $q = 0$ as a non-vanishing holomorphic function, of winding number 0 around small loops $|q| = \text{const}$.

Now shrink r so that V is holomorphic and non-vanishing on the simply connected domain V_r, and fix there a branch of

$$F_0 := \tfrac{1}{2\pi i}\log V, \qquad e^{2\pi i F_0} = V;$$

this is possible because V_r is simply connected and V is non-vanishing, equivalently because the winding number is 0. Put $F := F_0(q)$, a holomorphic function on U_r. Since $q = e^{2\pi i\zeta}$ is $\langle g_0\rangle$-invariant, F is g_0-invariant. Moreover $e^{2\pi i(\tau-\zeta)} = V(q) = e^{2\pi i F}$, so $(\tau-\zeta) - F$ is a continuous integer-valued function on the connected set U_r, hence a constant $k \in \mathbb{Z}$. Therefore $\tau - \zeta = F + k$ is single-valued and g_0-invariant:

$$(\tau-\zeta)\circ g_0 = (\tau-\zeta).$$

It is the single-valuedness of the chosen logarithm on the simply connected V_r that yields this; boundedness of $\mathrm{Im}(\tau-\zeta)$ alone would still allow a non-zero integer period. Consequently

$$\zeta\circ g_0 - \zeta = \tau\circ g_0 - \tau = -1, \qquad \text{i.e.} \quad \varepsilon_0 = -1.$$

The value of ε_0 is thus determined by the data rather than chosen. It is in any case not needed elsewhere: §3 uses only the $\langle g_0\rangle$-invariance of $q = e^{2\pi i\zeta}$ together with the quoted equivariance, and an asymptotic comparison of τ with ζ at the cusp, should one be needed, may be made with $\varepsilon_0 = -1$ and $\tau - \zeta$ bounded.

Remark 2.13. $\Delta \cong \mathbb{Z}/3 * \mathbb{Z}/4$ is the $(3,4,\infty)$ triangle group, not $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$; the modular group enters only through the orbifold morphism $(\mathbb{P}^1; 3,4,\infty) \to (\mathbb{P}^1; 3,2,\infty) = X(1)$, $t \mapsto 1728\,t$, which induces the surjection $\Delta \to PSL_2(\mathbb{Z})$ along which τ is equivariant in §3.

Definition 2.14 (Linearising coordinates)**.** For $j = 1,2$ let $s_j(z) := (z - z_j)/(z - \bar{z}_j)$ be the Cayley coordinate at z_j of Lemma 2.10, so $s_j(z_j) = 0$ and

$$s_j \circ g_j = \zeta_j\, s_j. \tag{2.4}$$

By proper discontinuity and Proposition 2.11(i) we may choose $r_j > 0$ so small that $\Delta_j := s_j^{-1}(\{|s| < r_j\})$ satisfies $g\Delta_j \cap \Delta_j = \emptyset$ for $g \in \Delta \setminus \langle g_j\rangle$. By (2.4) Δ_j is g_j-invariant and π induces a biholomorphism $\Delta_j/\langle g_j\rangle \xrightarrow{\sim} D_j := \pi(\Delta_j)$, an open neighbourhood of p_j; since $s_j^{m_j}$ is $\langle g_j\rangle$-invariant and induces $\Delta_j/\langle g_j\rangle \cong \{|s| < r_j^{m_j}\}$, there is a unique holomorphic t_j on D_j with $t_j \circ \pi = s_j^{m_j}$ on Δ_j, and t_j is a coordinate centred at p_j. Put $D_j^* := D_j \setminus \{p_j\}$, $\Delta_j^* := \Delta_j \setminus \{z_j\}$.

Definition 2.15 (The representation)**.** Since $\Delta = \langle g_1\rangle * \langle g_2\rangle$ with g_j of order m_j and $T_1^3 = T_2^4 = I$, the universal property of the free product gives a unique homomorphism

$$\rho_V\colon \Delta \to GL(V), \qquad \rho_V(g_1) = T_1, \quad \rho_V(g_2) = T_2,$$

with image G and $\rho_V(g_0) = T_0$. Put $\rho_\Lambda := \mathfrak{A}\circ\rho_V$, so $\rho_\Lambda(g_j) = A_j$, $\rho_\Lambda(g_0) = M_0$ (consistently with $A_1A_2M_0 = I$). For $g \in \Delta$ write $M_g := \rho_V(g)^t$ for the matrix acting on Λ-coordinates in §3; thus $g \mapsto M_g$ is an anti-homomorphism and $M_g^{-1} = \rho_\Lambda(g)$, so $M_{g_j}^{-1} = A_j$ and $M_{g_0}^{-1} = M_0$.

By Lemma 2.4 the functional γ is Δ-invariant for ρ_Λ, so the integers $\ell_j = \gamma(v_j)$ are well-defined invariants of the twisting data, used in §6 and §7; $\Lambda^\Delta = \mathbb{Z}\hat{\delta}$ by Lemma 2.7(ii) generates the vertical field ξ of §9, and Lemma 2.7(iv) is the input to the van Kampen computation of §7.

3 The period family over the thrice-punctured sphere.

Throughout this section we use the data and conventions of §2: $\Delta = \langle g_1, g_2 \mid g_1^3 = g_2^4 = 1\rangle$ is the $(3,4,\infty)$ triangle group, $\pi\colon \mathfrak{h}_z \to B \setminus \{p_0\}$ its uniformisation, $\rho_V(g_j) = T_j$ and $M_g := \rho_V(g)^t$; $\rho := e^{i\pi/3}$,

$\zeta_j := e^{-2\pi i/m_j}$ with $m_1 = 3$, $m_2 = 4$, and s_j is the linearising coordinate at z_j, $s_j \circ g_j = \zeta_j s_j$. By U_r we denote, as in §2, the *distinguished* connected cusp neighbourhood over $\{0 < |t_c| < r\}$: for r small, the connected components of $\pi^{-1}\big(\{0 < |t_c| < r\}\big)$ correspond to the cusps of Δ, each being stabilised by the corresponding conjugate of $\langle g_0\rangle$, and U_r is the unique $\langle g_0\rangle$-invariant one, i.e. the component at the fixed point of g_0 on $\partial\mathfrak{h}_z$. It is simply connected, $U_r/\langle g_0\rangle \cong \{0 < |t_c| < r\}$, and it is squeezed between two horodiscs at that cusp, though it is not itself a horodisc. The conditions $(\mu 2)$ and $(\beta 2)$ below are imposed on this one component U_r only; see Remark 3.2. The goal of the section is the following statement, proved in §§3.1–3.6.

Definition 3.1. Let $\tau\colon \mathfrak{h}_z \to \mathfrak{h}$, $\mu\colon \mathfrak{h}_z \to \mathbb{C}$ and $\beta\colon \mathfrak{h}_z \to \mathbb{C}$ be holomorphic. We consider the following conditions, in which the right-hand sides are evaluated at $z \in \mathfrak{h}_z$ and the left-hand sides at $g\cdot z$ (so e.g. $(\tau 2)$ reads $\tau(g_1 z) = (\tau(z)-1)/\tau(z)$).

- $(\tau 1)$ $j\big(\tau(z)\big) = 1728\, t\big(\pi(z)\big)$;
- $(\tau 2)$ $\tau \circ g_1 = \dfrac{\tau - 1}{\tau}$ and $\tau \circ g_2 = -\dfrac{1}{\tau}$ (hence $\tau \circ g_0 = \tau - 1$);
- $(\tau 3)$ $\tau(z_1) = \rho = e^{i\pi/3}$ and $\tau(z_2) = i$.
- $(\mu 1)$ $\mu \circ g_1 = \dfrac{1-\mu}{\tau}$ and $\mu \circ g_2 = 1 + \dfrac{\mu}{\tau}$ (hence $\mu \circ g_0 = \mu$);
- $(\mu 2)$ μ is bounded on the distinguished cusp neighbourhood U_r, for some $r > 0$.
- $(\beta 1)$ $\beta \circ g_1 = \beta + 2 - \dfrac{6(1-\mu)^2}{\tau}$ and $\beta \circ g_2 = \beta - 3 - \dfrac{6\mu^2}{\tau}$ (hence $\beta \circ g_0 = \beta + 1$);
- $(\beta 2)$ $\beta + \tau$ is bounded on the distinguished cusp neighbourhood U_r;
- $(\beta 3)$ $D := \operatorname{Im}\beta - \dfrac{6(\operatorname{Im}\mu)^2}{\operatorname{Im}\tau} < 0$ on $\mathfrak{h}_z$.

Remark 3.2. Conditions $(\mu 2)$, $(\beta 2)$ are imposed on the single component U_r fixed above rather than on all components of $\pi^{-1}(\{0 < |t_c| < r\})$; they are not Δ-equivariant. Indeed $(\mu 1)$ and $(\tau 2)$ give

$$\mu \circ g_2^2 = 1 + \frac{\mu \circ g_2}{\tau \circ g_2} = 1 - \tau\Big(1 + \frac{\mu}{\tau}\Big) = 1 - \tau - \mu,$$

which is unbounded on U_r (there $\operatorname{Im}\tau \to \infty$ by Corollary 3.7) as soon as μ is bounded there; equivalently, μ is bounded on U_r but not on $g_2^{-2}U_r$. The same remark applies to $\beta + \tau$.

Definition 3.3. Given τ, μ, β as above, set

$$\Pi(z) := \begin{pmatrix} 6\mu(z) & \tau(z) & 1 & 0 \\ \beta(z) & \mu(z) & 0 & 1 \end{pmatrix}, \qquad Z(z) := \begin{pmatrix} 6\mu(z) & \tau(z) \\ \beta(z) & \mu(z) \end{pmatrix},$$

so that $\Pi = [\, Z \mid I \,]$, the four columns of Π being indexed by the basis $(\hat\gamma, \hat u, \hat w, \hat\delta)$ of Λ; thus $\Pi(z)$ is the $\mathbb{C}$-linear map $\Lambda \otimes \mathbb{C} \to \mathbb{C}^2$ sending λ to $\Pi(z)\lambda$. Put

$$L(z) := \Pi(z)\Lambda \subset \mathbb{C}^2, \qquad \mathcal{T} := \big(\mathfrak{h}_z \times \mathbb{C}^2\big)/\Lambda, \quad \lambda\cdot(z,\zeta) := \big(z, \zeta + \Pi(z)\lambda\big),$$

so that the fibre of $\mathcal{T} \to \mathfrak{h}_z$ over z is $\mathbb{C}^2/L(z)$. For $g \in \Delta$ write the 2×2 block decomposition

$$[\, P_g(z) \mid R_g(z) \,] := \Pi(gz) M_g^{-1},$$

and $\tilde g(z,\zeta) := \big(gz, R_g(z)\zeta\big)$. Finally $\mathfrak{h}_z^\circ := \mathfrak{h}_z \smallsetminus (\Delta z_1 \cup \Delta z_2)$ and $J := \big(\mathcal{T}|_{\mathfrak{h}_z^\circ}\big)/\Delta \to B^\circ$.

Theorem 3.4.

(i) There is exactly one holomorphic $\tau\colon \mathfrak{h}_z \to \mathfrak{h}$ satisfying $(\tau 1)$ and $(\tau 2)$, and it satisfies $(\tau 3)$: $\tau(z_1) = \rho$, $\tau(z_2) = i$. Moreover $\tau - \rho$ vanishes to order 1 at z_1 and $\tau - i$ to order 2 at z_2, and on the cusp neighbourhood U_r one has $q = e^{2\pi i\tau} = t_c\, u_1(t_c)$ with u_1 holomorphic and nowhere zero on $\{|t_c| < r\}$.

(ii) There is exactly one holomorphic $\mu\colon \mathfrak{h}_z \to \mathbb{C}$ satisfying ($\mu 1$) and ($\mu 2$). It satisfies $\mu(z_1) = \frac{1}{1+\rho} = \frac{2-\rho}{3}$ and $\mu(z_2) = \frac{1-i}{2}$, hence

$$6\big(1-\mu(z_1)\big)^2 = 2\tau(z_1), \qquad 6\mu(z_2)^2 = -3\tau(z_2).$$

As a function of t_c near p_0 (computed on U_r), μ extends holomorphically across $t_c = 0$.

(iii) The set of holomorphic $\beta\colon \mathfrak{h}_z \to \mathbb{C}$ satisfying ($\beta 1$) and ($\beta 2$) is non-empty and equals $\{\beta^{\mathrm{part}} + c_0 : c_0 \in \mathbb{C}\}$ for any one solution β^{part}. The function $b := \beta + \tau$ is g_0-invariant and, as a function of t_c on U_r, extends holomorphically across $t_c = 0$.

(iv) The function $D = \operatorname{Im}\beta - 6(\operatorname{Im}\mu)^2/\operatorname{Im}\tau$ is Δ-invariant; the induced function on $B \smallsetminus \{p_0\}$ is continuous (in particular at p_1, p_2) and tends to $-\infty$ at p_0. Consequently there is $M \in \mathbb{R}$ such that every c_0 with $\operatorname{Im} c_0 < -M$ makes ($\beta 3$) hold. Let β be such a function. Then $L(z) = \Pi(z)\Lambda$ is a lattice in $\mathbb{C}^2$ for *every* $z \in \mathfrak{h}_z$, including $z = z_1, z_2$, and $\mathcal{T} \to \mathfrak{h}_z$ is a holomorphic family of compact complex 2-tori.

(v) For every $g \in \Delta$ the matrix $R_g(z)$ (the right block of $\Pi(gz)M_g^{-1}$) is invertible, $\Pi(gz) = R_g(z)\Pi(z)M_g$ and $R_{gh}(z) = R_g(hz)R_h(z)$; $R_{g_0} \equiv I$. Hence $g \mapsto \tilde{g}$ is an action of Δ on $\mathcal{T}$ by fibre-preserving biholomorphisms, free on $\mathcal{T}|_{\mathfrak{h}_z^\circ}$, and $J = (\mathcal{T}|_{\mathfrak{h}_z^\circ})/\Delta \to B^\circ$ is a proper holomorphic submersion with fibres complex 2-tori and a holomorphic zero section. With the marking and the orientation conventions of §3.6 the monodromy representation on $H_1 = \Lambda$ (in the inverse-transport convention of §7; see the proof of Proposition 3.19) is $g \mapsto M_g^{-1}$; the loops lifting to g_1, g_2, g_0 act by A_1, A_2, M_0.

Theorem 3.4 makes no positivity (polarisation) assertion; Remark 3.23 computes the relevant Hermitian form and finds it of signature $(1,1)$.

3.1 The uniformising function τ

We use the following classical facts about the modular function j, normalised as in §2, and the forms $E_4, E_6, \Delta_{\mathrm{mod}}$ of weights $4, 6, 12$ [Se73]: j induces a biholomorphism $\mathfrak{h}/PSL_2(\mathbb{Z}) \to \mathbb{C}$, and over $\mathbb{C} \smallsetminus \{0, 1728\}$ it restricts to a *regular* (Galois) unramified covering

$$j\colon \mathfrak{h} \smallsetminus \big(PSL_2(\mathbb{Z})\rho \cup PSL_2(\mathbb{Z})i\big) \longrightarrow \mathbb{C} \smallsetminus \{0, 1728\}$$

with deck group

$$PSL_2(\mathbb{Z}) = \langle S\rangle * \langle S_1\rangle \cong \mathbb{Z}/2 * \mathbb{Z}/3, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ S_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix};$$

under the resulting surjection $\pi_1(\mathbb{C} \smallsetminus \{0, 1728\}) \twoheadrightarrow PSL_2(\mathbb{Z})$ a meridian of 0 maps to an element of order 3, conjugate to $S_1^{\pm 1}$, and a meridian of 1728 to an element of order 2, conjugate to S. (Only the orders 3 and 2 of these two elements will be used, and they are unchanged under inversion, so either orientation of the two meridians serves here; the orientation conventions that matter for the marking are those of Convention 3.17.) Furthermore j has a zero of order 3 on the orbit of ρ, and $j - 1728$ a zero of order 2 on the orbit of i; E_4 vanishes exactly and simply on the ρ-orbit, E_6 exactly and simply on the i-orbit, Δ_{mod} is nowhere zero and $\Delta_{\mathrm{mod}} = q + O(q^2)$, $q = e^{2\pi i\tau}$. We write $j = J(q)$, where

$$J(q) = q^{-1}\big(1 + O(q)\big)$$

is holomorphic on $\{0 < |q| < 1\}$ with a simple pole at $q = 0$.

We also use the following description of the region where $|j|$ is large.

Lemma 3.5. For $R > 1728$ put $\Omega_R := \{\tau \in \mathfrak{h} : |j(\tau)| > R\}$ and

$$c_1(R) := \inf\{\operatorname{Im}\tau : \tau \in \mathcal{F},\ |j(\tau)| > R\},$$

$\mathcal{F}$ the standard fundamental domain; then $c_1(R) \to \infty$ as $R \to \infty$. Fix once and for all $R_0 > 1728$ such that $c_1(R) > 1$ for all $R \geq R_0$.

Let $R \geq R_0$. Then Ω_R is $PSL_2(\mathbb{Z})$-invariant and $PSL_2(\mathbb{Z})$ permutes its connected components transitively. Let V_R be the component containing $\{\operatorname{Im}\tau > c\}$ for $c \gg 0$. There is $c_2(R) \geq c_1(R)$ with

$$\{\operatorname{Im}\tau > c_2(R)\} \ \subset\ V_R \ \subset\ \{\operatorname{Im}\tau \geq c_1(R)\},$$

i.e. V_R is squeezed between two horoballs at the cusp ∞ (it need not be a horoball). Consequently the components of Ω_R are the pairwise disjoint translates γV_R, $\gamma \in PSL_2(\mathbb{Z})$, each squeezed between two horoballs at the cusp $\gamma\infty$, and every $\gamma' \in PSL_2(\mathbb{Z})$ with $\gamma'\gamma V_R = \gamma V_R$ fixes the cusp $\gamma\infty$.

Proof. First, $c_1(R) \to \infty$: given $A > 0$, the set $\mathcal{F} \cap \{\operatorname{Im}\tau \le A\}$ is compact, so $|j|$ is bounded on it by some $R(A)$; for $R \ge R(A)$ no point of $\mathcal{F}$ with $\operatorname{Im}\tau \le A$ satisfies $|j| > R$, i.e. $c_1(R) \ge A$. In particular R_0 exists.

Now let $R \ge R_0$. $\Omega_R = j^{-1}(\{|w| > R\})$ is invariant, and j induces a biholomorphism $\Omega_R/PSL_2(\mathbb{Z}) \cong \{|w| > R\}$, a connected set; hence the components of Ω_R form one orbit. Since $j = q^{-1}(1 + O(q))$ as $\operatorname{Im}\tau \to \infty$, we have $|j| > R$ on $\{\operatorname{Im}\tau > c_2(R)\}$ for suitable $c_2(R) \ge c_1(R)$, so V_R is well defined.

For the other inclusion: by the definition of $c_1(R)$, $\{|j| > R\} \cap \mathcal{F} \subset \{\operatorname{Im}\tau \ge c_1(R)\}$; as every point of $\mathfrak{h}$ is $PSL_2(\mathbb{Z})$-equivalent to a point of $\mathcal{F}$ and Ω_R is invariant, this gives

$$\Omega_R \subset \bigcup_{\gamma \in PSL_2(\mathbb{Z})} \gamma\big(\{\operatorname{Im}\tau \ge c_1(R)\}\big),$$

a union of horoballs indexed by the cusps $\gamma\infty$. Since $R \ge R_0$ we have $c_1 := c_1(R) > 1$, and then these horoballs are pairwise disjoint (Shimizu's lemma for $PSL_2(\mathbb{Z})$: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has $c \ne 0$ then $\operatorname{Im}(\gamma\tau) = \operatorname{Im}\tau/|c\tau + d|^2 \le 1/\operatorname{Im}\tau$, so $\operatorname{Im}\tau \ge c_1 > 1$ and $\operatorname{Im}(\gamma\tau) \ge c_1 > 1$ would force $1 < c_1 \le \operatorname{Im}(\gamma\tau) \le 1/\operatorname{Im}\tau \le 1/c_1 < 1$; hence no point of height $\ge c_1$ is carried to a point of height $\ge c_1$ unless γ fixes ∞). Fix c_1' with $1 < c_1' < c_1$; the same displayed bound shows that the *open* horoballs $\gamma(\{\operatorname{Im}\tau > c_1'\})$ are pairwise disjoint as well, and they contain the closed ones above, so Ω_R is contained in a disjoint union of *open* horoballs, one at each cusp. These open sets induce a partition of Ω_R into relatively open pieces, so each connected subset of Ω_R lies in a single one of them, hence in a single closed horoball $\gamma(\{\operatorname{Im}\tau \ge c_1\})$. Since V_R is connected and contains $\{\operatorname{Im}\tau > c_2(R)\}$, the horoball in question is the one at ∞, i.e. $V_R \subset \{\operatorname{Im}\tau \ge c_1(R)\}$.

Finally, γV_R lies in a horoball at $\gamma\infty$, so its closure in $\mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$ meets $\mathbb{P}^1(\mathbb{Q})$ exactly in $\gamma\infty$; hence any γ' with $\gamma'\gamma V_R = \gamma V_R$ fixes $\gamma\infty$. □

Proposition 3.6. Part (i) of Theorem 3.4 holds.

Proof. Existence of a lift. Put $F := 1728\,(t \circ \pi)\colon \mathfrak{h}_z \to \mathbb{C}$. Since π is unramified outside $\pi^{-1}\{p_1, p_2\}$ and ramified there of order exactly 3, resp. 4, the map F (resp. $F - 1728$) vanishes to order exactly 3 (resp. 4) on $\pi^{-1}(p_1)$ (resp. $\pi^{-1}(p_2)$), and F has no other critical points. Let $Z^\circ := \mathfrak{h}_z \smallsetminus F^{-1}\{0, 1728\}$; its fundamental group is normally generated by small meridians (of either orientation, since only the orders below are used), whose F-images are cubes (resp. fourth powers) of meridians of 0 (resp. 1728), whose classes in the deck group $PSL_2(\mathbb{Z})$ have order 3 (resp. 2) and hence are trivial. Therefore the image of $F_*\pi_1(Z^\circ)$ in the deck group is trivial, i.e. $F_*\pi_1(Z^\circ) \subset j_*\pi_1\big(\mathfrak{h} \smallsetminus (PSL_2(\mathbb{Z})\rho \cup PSL_2(\mathbb{Z})i)\big)$, which is exactly the lifting criterion for the covering j. So $F|_{Z^\circ}$ lifts to a holomorphic $\tau\colon Z^\circ \to \mathfrak{h}$ with $j \circ \tau = F$.

We extend τ across each point $z_0 \in F^{-1}\{0, 1728\}$ by the maximum principle in the bounded model. Let $\delta\colon \mathfrak{h} \to \mathbb{D}$ be a biholomorphism onto the unit disc and let D^* be a small punctured disc around z_0 contained in Z° (possible, as $F^{-1}\{0, 1728\}$ is discrete). Then $\delta \circ \tau$ is a bounded holomorphic function on D^*, so it extends holomorphically across z_0, with $|(\delta \circ \tau)(z_0)| \le 1$. It is non-constant, because $j \circ \tau = F$ is non-constant near z_0; hence the maximum principle excludes $|(\delta \circ \tau)(z_0)| = 1$, and the extension takes values in $\mathbb{D}$. Thus τ extends holomorphically across z_0 with value in $\mathfrak{h}$, and $j \circ \tau = F$ persists by continuity: this is $(\tau 1)$.

Equivariance. $F \circ g = F$, so $\tau \circ g$ and τ are lifts of the same map through the Galois covering j: $\tau \circ g = \varphi(g) \circ \tau$ with $\varphi\colon \Delta \to PSL_2(\mathbb{Z})$ a homomorphism.

Local orders. Near z_1, $t \circ \pi = s_1^3 \cdot (\text{unit})$ while j has a zero of order 3 at ρ, so $\tau - \tau(z_1)$ has a simple zero; near z_2, $(t-1) \circ \pi = s_2^4 \cdot (\text{unit})$ and $j - 1728$ has a zero of order 2, so $\tau - \tau(z_2)$ has a zero of order exactly 2.

Normalisation. Replacing τ by $\gamma \circ \tau$ replaces φ by $\gamma\varphi\gamma^{-1}$; since $j(\tau(z_1)) = 0$ we may arrange $\tau(z_1) = \rho$. Then $\varphi(g_1)$ fixes ρ and, τ being a local biholomorphism at z_1, $\varphi(g_1)'(\rho) = g_1'(z_1) = e^{-2\pi i/3}$; as the stabiliser of ρ is $\langle S_1 \rangle$ and $S_1'(\rho) = \rho^{-2} = e^{-2\pi i/3}$, $\varphi(g_1) = S_1$.

Rigidity: the order of $\varphi(g_2)$. Put $P := \varphi(g_0)$ and $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. We claim $\varphi(g_2) \neq 1$. Otherwise $\tau \circ g_2 = \tau$, so the holomorphic function $\tau - \tau(z_2)$ is g_2-invariant near z_2; in the linearising coordinate s_2 (with $s_2 \circ g_2 = \zeta_2 s_2$, $\zeta_2 = -i$ a primitive 4th root of unity) a g_2-invariant holomorphic function has only exponents in $4\mathbb{Z}$, whereas $\operatorname{ord}_{z_2}\big(\tau - \tau(z_2)\big) = 2 \notin 4\mathbb{Z}$ by the computation of local orders. Since $\varphi(g_2)^4 = 1$ and $PSL_2(\mathbb{Z})$ has no element of order 4, $\varphi(g_2)$ has order 2. As $\varphi(g_2) = \varphi(g_1)^{-1}P^{-1} = S_1^{-1}P^{-1}$, this reads $\operatorname{tr}\big(S_1^{-1}P^{-1}\big) = 0$ for either SL_2-lift.

Rigidity: P is parabolic. First $P \neq 1$. Otherwise $\tau \circ g_0 = \tau$, so τ is g_0-invariant on U_r and descends to a holomorphic function of t_c on $\{0 < |t_c| < r\}$; so does $q = e^{2\pi i \tau}$, which moreover satisfies $|q| < 1$ (as $\operatorname{Im}\tau > 0$) and hence extends holomorphically across $t_c = 0$ with $|q(0)| \leq 1$. If $|q(0)| = 1$, then $|q|$ would attain its maximum at the interior point $t_c = 0$, so q would be a constant of modulus 1, contradicting $|q| < 1$; hence $|q(0)| < 1$. Next, $j(\tau) = J(q)$ with J holomorphic on $\{0 < |q| < 1\}$ and with a simple pole at $q = 0$. If $q(0) \neq 0$ then $j(\tau) = J\big(q(t_c)\big) \to J\big(q(0)\big) \in \mathbb{C}$ as $t_c \to 0$, contradicting $|j(\tau)| = 1728|t_c|^{-1} \to \infty$. Hence $q(0) = 0$; put $n := \operatorname{ord}_0 q \geq 1$, so that the winding number of q along a small circle $|t_c| = \varepsilon$ is $n \geq 1$. On the other hand $2\pi i\tau$ is, by g_0-invariance, a single-valued continuous (indeed holomorphic) logarithm of q on $\{0 < |t_c| < r\}$, so that winding number is 0 — a contradiction. Thus $P \neq 1$.

Next, shrink r so that $R := 1728/r \geq R_0$, with R_0 as in Lemma 3.5. Then $|j(\tau)| = 1728/|t_c| > R$ on U_r, i.e. $\tau(U_r) \subset \Omega_R$, and U_r is connected, so by Lemma 3.5 $\tau(U_r)$ lies in a single connected component γV_R of Ω_R. Since $g_0 U_r = U_r$ and $\tau \circ g_0 = P \circ \tau$, the set $\tau(U_r)$ is $\langle P\rangle$-invariant, so $P(\gamma V_R) \cap \gamma V_R \neq \emptyset$ and therefore $P(\gamma V_R) = \gamma V_R$ by disjointness of the components. By Lemma 3.5, P fixes the cusp $\gamma\infty$; the stabiliser of a cusp in $PSL_2(\mathbb{Z})$ is infinite cyclic generated by a parabolic element, so P is parabolic.

Rigidity: conclusion. Write the $SL_2(\mathbb{Z})$-lift as $P = I + M$ with M nilpotent,

$$M = k\begin{pmatrix} -pr & p^2 \\ -r^2 & pr \end{pmatrix}, \qquad \gcd(p, r) = 1,\ k \in \mathbb{Z} \smallsetminus \{0\}.$$

With $S_1^{-1} = \left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right)$ one computes $\operatorname{tr}\big(S_1^{-1}(I - M)\big) = 1 - (c - b - a)$ for $M = \left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)$, so the condition $\operatorname{tr}(S_1^{-1}P^{-1}) = 0$ reads $c - b - a = 1$, i.e. $-k(p^2 - pr + r^2) = 1$. By positive definiteness $k = -1$ and $p^2 - pr + r^2 = 1$, with solutions $\pm(1,0), \pm(0,1), \pm(1,1)$; the corresponding P are T^{-1} and its two S_1-conjugates (the cusps $\infty, 0, 1$ form one $\langle S_1\rangle$-orbit). Conjugating τ by a power of S_1 (which preserves $\tau(z_1) = \rho$ and $\varphi(g_1) = S_1$) gives $\varphi(g_0) = T^{-1}$, hence $\tau \circ g_0 = \tau - 1$ and $\varphi(g_2) = S_1^{-1}T = S$, i.e. $\tau \circ g_2 = -1/\tau$: this is (τ2), and $\tau(z_2)$, the fixed point of S, equals i, which is (τ3).

Uniqueness. If τ' satisfies (τ1), (τ2) then $\tau' = \gamma \circ \tau$ with $\gamma \in PSL_2(\mathbb{Z})$; equivariance for both gives $\gamma\varphi(g)\gamma^{-1} = \varphi(g)$ for $g = g_1, g_2$, and $\langle S_1, S\rangle = PSL_2(\mathbb{Z})$ has trivial centre, so $\gamma = 1$.

The cusp. By (τ2), $\tau \circ g_0 = \tau - 1$, so on U_r the function $q = e^{2\pi i\tau}$ is g_0-invariant, hence a holomorphic function of t_c on $\{0 < |t_c| < r\}$ with $|q| < 1$; it extends holomorphically across 0, and exactly as in the paragraph "P is parabolic" the maximum principle gives $|q(0)| < 1$, while the simple pole of J at $q = 0$, together with $|j(\tau)| = 1728|t_c|^{-1} \to \infty$, forces $q(0) = 0$. From $j = q^{-1}(1 + O(q))$ and $j(\tau) = 1728/t_c$ we get $q = \frac{t_c}{1728}(1 + O(t_c))$, so $q = t_c\, u_1(t_c)$ with u_1 holomorphic and nowhere zero after shrinking r. $\square$

Corollary 3.7. *Let $h := \frac{1}{2\pi i}\log u_1$ (a branch on $\{|t_c| < r\}$) and $s := \tau - h(t_c \circ \pi)$ on U_r. Then $e^{2\pi i s} = t_c \circ \pi$, $s \circ g_0 = s - 1$, and s maps U_r biholomorphically onto a half plane $\{\operatorname{Im} s > \text{const}\}$. In particular $\operatorname{Im}\tau \to \infty$ as $t_c \to 0$ on U_r.*

Proof. $e^{2\pi i s} = q/u_1 = t_c \circ \pi$, and $h(t_c \circ \pi)$ is g_0-invariant, so $s \circ g_0 = s - 1$. Both $t_c \circ \pi \colon U_r \to \{0 < |t_c| < r\}$ and $e^{2\pi i(\cdot)}$ are universal coverings of the punctured disc with deck group $\mathbb{Z}$, so s is an isomorphism of coverings; $\tau = s + h$ with h bounded gives the last assertion. $\square$

3.2 The function μ: a torsor under a line bundle of degree -1

Define $\tilde{\jmath} \colon \Delta \times \mathfrak{h}_z \to \mathbb{C}^*$ by $\tilde{\jmath}(g_1, z) := -\tau(z)$, $\tilde{\jmath}(g_2, z) := \tau(z)$ and the cocycle rule $\tilde{\jmath}(gh, z) = \tilde{\jmath}(g, hz)\tilde{\jmath}(h, z)$.

Lemma 3.8. *$\tilde{\jmath}$ is a well-defined 1-cocycle on Δ, and $\tilde{\jmath}(g_0, \cdot) \equiv 1$.*

Proof. As $\Delta = \langle g_1 \rangle * \langle g_2 \rangle$, a unique such cocycle exists provided that $\prod_{k=0}^{m_j-1} \tilde{\jmath}(g_j, g_j^k z) = 1$ for $j = 1, 2$. By $(\tau 2)$ one has $\tau \circ g_1 = (\tau - 1)/\tau$, $\tau \circ g_1^2 = -1/(\tau - 1)$ and $\tau \circ g_2 = -1/\tau$; direct computation then gives

$$(-1)^3\, \tau \cdot \frac{\tau - 1}{\tau} \cdot \frac{-1}{\tau - 1} = 1,$$
$$\tau \cdot \Big(-\frac{1}{\tau}\Big) \cdot \tau \cdot \Big(-\frac{1}{\tau}\Big) = 1.$$

Finally $\tilde{\jmath}(g_1 g_2, z) = -\tau(g_2 z)\tau(z) = 1$, so $\tilde{\jmath}(g_0, \cdot) \equiv 1$. $\square$

Definition 3.9. For $U \subset B$ open put $\widetilde{U} := \pi^{-1}(U \smallsetminus \{p_0\})$ and

$$\mathfrak{L}(U) := \big\{\nu \in \mathcal{O}(\widetilde{U}) : \nu \circ g = \nu / \tilde{\jmath}(g, \cdot)\ \forall g \in \Delta\big\},$$
$$\mathfrak{M}(U) := \big\{\mu \in \mathcal{O}(\widetilde{U}) : (\mu 1) \text{ holds}\big\},$$

where, if $p_0 \in U$, we require in addition the following regularity *at the distinguished cusp*: shrinking U we may assume $U \subset \{|t_c| < r\}$, and then $U_r(U) := U_r \cap \widetilde{U}$ is the $\langle g_0 \rangle$-invariant component of $\widetilde{U}$ over $U \smallsetminus \{p_0\}$; the restriction of the function to $U_r(U)$ is $\langle g_0 \rangle$-invariant, hence descends to a holomorphic function of t_c on $U \smallsetminus \{p_0\}$, and we require that this function extend holomorphically across $t_c = 0$.

The condition just imposed is local in U and stable under restriction, so $\mathfrak{L}$ and $\mathfrak{M}$ are sheaves, of $\mathcal{O}_B$-modules resp. of sets. It concerns the one component $U_r(U)$, and is strictly weaker than the corresponding condition on all components (Remark 3.2); this is exactly the asymmetry of $(\mu 2)$, $(\beta 2)$. Note also that $\nu \in \mathfrak{L}(U)$ is g_0-invariant by Lemma 3.8, and $\mu \in \mathfrak{M}(U)$ is g_0-invariant by $(\mu 1)$, so the descent used in the definition makes sense. Finally $\mathfrak{L}$ acts on $\mathfrak{M}$ by addition: the difference $\nu = \mu - \mu'$ of two sections of $\mathfrak{M}(U)$ satisfies $\nu \circ g_1 = -\nu/\tau$, $\nu \circ g_2 = \nu/\tau$.

Lemma 3.10. $\mathfrak{L} \cong \mathcal{O}_B(-p_0) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$; in particular $H^0(B, \mathfrak{L}) = H^1(B, \mathfrak{L}) = 0$.

Proof. Since $\mathfrak{h}_z$ is simply connected and $E_6 \circ \tau$ vanishes exactly on Δz_2, to order 2 at each point, it has a holomorphic square root on $\mathfrak{h}_z$; fix one and set $F := \big(E_4^2 E_6^{1/2}/\Delta_{\mathrm{mod}}\big) \circ \tau$. Then F is holomorphic and vanishes exactly on $\Delta z_1 \cup \Delta z_2$, with $\operatorname{ord}_{z_1} F = 2$, $\operatorname{ord}_{z_2} F = 1$; near the cusp $E_4, E_6 \to 1$ and $\Delta_{\mathrm{mod}} = q + O(q^2)$, so $F = t_c^{-1} \cdot (\text{unit})$ as a function of t_c on U_r.

F obeys the homogeneous law: by modularity of weights $8, 3, 12$ (computed with $SL_2(\mathbb{Z})$-lifts of the images of g_1, g_2 in $PSL_2(\mathbb{Z})$; changing a lift changes the automorphy factor by the sign $(-1)^{8+3-12} = -1$, which is absorbed in $\chi(g_j)$; a definite lift is fixed in Lemma 9.14(ii)), $F \circ g_j = \chi(g_j) F / \tilde{\jmath}(g_j, \cdot)$ with $\chi(g_j) = \pm 1$ (the ambiguity comes only from the square root) and $\chi(g_j)^{m_j} = 1$ by iterating m_j times and using Lemma 3.8; hence $\chi(g_1) = 1$. For g_2 evaluate at z_2: $E_6^{1/2} \circ \tau = c\, s_2 + O(s_2^2)$ with $c \neq 0$ and $s_2 \circ g_2 = -i s_2$, so the value at $g_2 z$ is $-i(E_6^{1/2} \circ \tau)(z) + O(s_2^2)$, whereas the claimed law gives the factor $\chi(g_2)\tau(z_2)^3 = -i\chi(g_2)$; so $\chi(g_2) = 1$ and $F \in \mathfrak{L}(B \smallsetminus \{p_0\})$, with $t_c F$ nowhere zero at p_0.

F generates $\mathfrak{L}$ away from p_0: for $\nu \in \mathfrak{L}_p$ the quotient ν/F is Δ-invariant and meromorphic, hence a meromorphic function of t; it is holomorphic, since at p_1 its pole order is at most 2 in s_1, i.e. at most $2/3$ in $t_1 = s_1^3$, and at p_2 at most 1 in s_2, i.e. at most $1/4$ in $t_2 = s_2^4$. Conversely $\psi F \in \mathfrak{L}_p$ for $\psi \in \mathcal{O}_{B,p}$. At p_0, $\nu = \psi F$ lies in $\mathfrak{L}_{p_0}$ iff ψ vanishes, i.e. $\mathfrak{L}_{p_0} = \mathfrak{m}_{p_0} F$. Altogether $\mathfrak{L} \cong \mathcal{O}_B(-p_0)$. $\square$

Proposition 3.11. $\mathfrak{M}$ is an $\mathfrak{L}$-torsor, and part (ii) of Theorem 3.4 holds.

Proof. Consistency of the laws $(\mu 1)$ *with* $g_1^3 = g_2^4 = 1$. Since $\Delta = \langle g_1 \rangle * \langle g_2 \rangle$ is the free product of cyclic groups of orders 3 and 4, prescribing μ on one sheet (or on a g_j-invariant disc) and extending by $(\mu 1)$ along words in g_1, g_2 is well defined if and only if the affine substitutions occurring in $(\mu 1)$ have orders dividing 3, resp. 4, along the corresponding τ-orbits. This holds. Indeed, by $(\tau 2)$ and $(\mu 1)$,

	z	$g_1 z$	$g_1^2 z$
τ	τ	$\dfrac{\tau - 1}{\tau}$	$\dfrac{-1}{\tau - 1}$
μ	μ	$\dfrac{1 - \mu}{\tau}$	$\dfrac{\tau - 1 + \mu}{\tau - 1}$

	z	$g_2 z$	$g_2^2 z$	$g_2^3 z$
τ	τ	$-\dfrac{1}{\tau}$	τ	$-\dfrac{1}{\tau}$
μ	μ	$1 + \dfrac{\mu}{\tau}$	$1 - \tau - \mu$	$\dfrac{1 - \mu}{\tau}$

(each entry obtained from its left neighbour by $(\mu 1)$), and one more application of $(\mu 1)$ gives

$$\frac{1-\frac{\tau-1+\mu}{\tau-1}}{\frac{-1}{\tau-1}} = -\big((\tau-1)-(\tau-1+\mu)\big) = \mu, \qquad 1+\frac{(1-\mu)/\tau}{-1/\tau} = \mu,$$

i.e. the prescriptions close up after 3, resp. 4, steps, as required. (Equivalently: $(\mu 1)$ is an affine cocycle for $\tilde{\jmath}$, and the multiplicative parts close up by Lemma 3.8.)

Local sections. At $p \in B^\circ$: for small U, $\pi^{-1}(U)$ is a disjoint union of sheets permuted simply transitively by Δ; prescribe $\mu := 0$ on one sheet and extend by the laws $(\mu 1)$, which is consistent by the previous paragraph. At p_0: $\mu := 0$ on U_r is g_0-invariant and holomorphic at $t_c = 0$ (and extends to $\widetilde{U}$ by the laws). At p_1, p_2 take $\mu_1 := (2-\tau)/3$ on Δ_1 and $\mu_2 := (1-\tau)/2$ on Δ_2 and extend by equivariance; indeed, by direct computation,

$$\mu_1 \circ g_1 = \tfrac{\tau+1}{3\tau} = (1-\mu_1)/\tau, \qquad \mu_2 \circ g_2 = \tfrac{\tau+1}{2\tau} = 1+\mu_2/\tau.$$

Global sections. By Riemann's theorem applied to the bounded g_0-invariant function μ of t_c on U_r, a holomorphic μ on $\mathfrak{h}_z$ satisfies $(\mu 1)$, $(\mu 2)$ exactly when it lies in $\mathfrak{M}(B)$. The torsor $\mathfrak{M}$ is classified by $H^1(B, \mathfrak{L}) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, so $\mathfrak{M}(B) \neq \emptyset$, and two global sections differ by an element of $H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$.

Values. Evaluating $(\mu 1)$ at the fixed points gives $\mu(z_1) = (1-\mu(z_1))/\rho$, i.e. $\mu(z_1) = \frac{1}{1+\rho} = \frac{2-\rho}{3}$ (using $\rho^2 = \rho - 1$), and $\mu(z_2) = 1 + \mu(z_2)/i$, i.e. $\mu(z_2) = \frac{1-i}{2}$. Then $1-\mu(z_1) = \rho/(1+\rho)$ and $(1+\rho)^2 = 3\rho$ give $6(1-\mu(z_1))^2 = 2\rho = 2\tau(z_1)$, while $6\mu(z_2)^2 = -3i = -3\tau(z_2)$. □

These identities say that the inhomogeneous terms $2-6(1-\mu)^2/\tau$ and $-3-6\mu^2/\tau$ of $(\beta 1)$ vanish at z_1, resp. z_2; this makes the β-problem locally solvable at p_1, p_2.

Remark 3.12. We record a geometric interpretation, which will not be needed later. Condition $(\tau 1)$ says that τ is the period map of the rational elliptic surface with $j = 1728t$ and Weierstrass model $y^2 = 4x^3 - 3t^3(t-1)x - t^4(t-1)^2$, with fibres of types IV^*, III, I_1 over p_1, p_2, p_0 [Kod64], [BHPV, V.10]; its Mordell–Weil group is generated by $P = (0, i\,t^2(t-1))$, and $-\mu$ is the Abel–Jacobi coordinate of P, the values $\mu(z_1), \mu(z_2)$ being the 3- resp. 2-torsion values dictated by the non-identity components of the IV^* and III fibres.

3.3 The function β

Write $\phi_1 := 2-6(1-\mu)^2/\tau$ and $\phi_2 := -3-6\mu^2/\tau$, holomorphic on $\mathfrak{h}_z$ (recall $\tau \in \mathfrak{h}$), so that $(\beta 1)$ reads $\beta \circ g_j = \beta + \phi_j$.

Proposition 3.13. $\sum_{k=0}^{m_j-1} \phi_j \circ g_j^k = 0$ on $\mathfrak{h}_z$ for $j = 1, 2$, and part (iii) of Theorem 3.4 holds.

Proof. Cocycle relation. This is a direct computation with $(\tau 2)$ and $(\mu 1)$ only, using the two tables of the proof of Proposition 3.11, which list the values of τ and μ at $z, g_1 z, g_1^2 z$, resp. at $z, g_2 z, g_2^2 z, g_2^3 z$; all identities below are between holomorphic functions on $\mathfrak{h}_z$ (equivalently, between rational expressions in the two independent quantities τ, μ, valid wherever the denominators $\tau, \tau-1$ do not vanish, hence everywhere by continuity).

For $j = 1$: substituting $(\tau, \mu) \mapsto \big(\frac{\tau-1}{\tau}, \frac{1-\mu}{\tau}\big)$, resp. $(\tau, \mu) \mapsto \big(\frac{-1}{\tau-1}, \frac{\tau-1+\mu}{\tau-1}\big)$, into $\phi_1 = 2-6(1-\mu)^2/\tau$ gives

$$\begin{aligned}
\phi_1 &= 2 - \frac{6(1-\mu)^2}{\tau},\\
\phi_1 \circ g_1 &= 2 - 6\Big(1-\frac{1-\mu}{\tau}\Big)^2 \frac{\tau}{\tau-1} = 2 - 6\Big(\frac{\tau-1+\mu}{\tau}\Big)^2 \frac{\tau}{\tau-1} = 2 - \frac{6(\tau-1+\mu)^2}{\tau(\tau-1)},\\
\phi_1 \circ g_1^2 &= 2 - 6\Big(1-\frac{\tau-1+\mu}{\tau-1}\Big)^2 \big(-(\tau-1)\big) = 2 + 6(\tau-1)\frac{\mu^2}{(\tau-1)^2} = 2 + \frac{6\mu^2}{\tau-1}.
\end{aligned}$$

Hence
$$\sum_{k=0}^{2}\phi_1\circ g_1^k = 6-\frac{6}{\tau(\tau-1)}\Big[(1-\mu)^2(\tau-1)+(\tau-1+\mu)^2-\mu^2\tau\Big].$$
Writing $a:=\tau-1$, the bracket is
$$a(1-2\mu+\mu^2)+(a+\mu)^2-\mu^2(a+1) = a-2a\mu+a\mu^2+a^2+2a\mu+\mu^2-a\mu^2-\mu^2 = a+a^2 = \tau(\tau-1),$$
so the sum equals $6-6=0$.

For $j=2$: substituting successively $(\tau,\mu)\mapsto\big(-\frac{1}{\tau},1+\frac{\mu}{\tau}\big)$, $(\tau,1-\tau-\mu)$, $\big(-\frac{1}{\tau},\frac{1-\mu}{\tau}\big)$ into $\phi_2=-3-6\mu^2/\tau$ gives
$$\phi_2 = -3-\frac{6\mu^2}{\tau}, \qquad \phi_2\circ g_2 = -3+6\tau\Big(1+\frac{\mu}{\tau}\Big)^2 = -3+\frac{6(\tau+\mu)^2}{\tau},$$
$$\phi_2\circ g_2^2 = -3-\frac{6(1-\tau-\mu)^2}{\tau}, \qquad \phi_2\circ g_2^3 = -3+6\tau\frac{(1-\mu)^2}{\tau^2} = -3+\frac{6(1-\mu)^2}{\tau}.$$
Hence
$$\sum_{k=0}^{3}\phi_2\circ g_2^k = -12+\frac{6}{\tau}\Big[-\mu^2+(\tau+\mu)^2-(1-\tau-\mu)^2+(1-\mu)^2\Big].$$
Now $(\tau+\mu)^2-\mu^2=\tau^2+2\tau\mu$, and, with $A:=1-\mu$ so that $1-\tau-\mu=A-\tau$, $(1-\mu)^2-(1-\tau-\mu)^2 = A^2-(A-\tau)^2 = 2A\tau-\tau^2 = 2\tau(1-\mu)-\tau^2$; the bracket is therefore $\tau^2+2\tau\mu+2\tau-2\tau\mu-\tau^2=2\tau$, and the sum equals $-12+12=0$.

(In particular the vanishing of these sums is, for the β-laws, the analogue of the consistency check carried out for $(\mu1)$ in the proof of Proposition 3.11: since $\Delta=\langle g_1\rangle * \langle g_2\rangle$ with g_j of order m_j, it is exactly what makes "prescribe β and extend by $(\beta1)$" well defined.)

Torsor. Since $\beta\circ g_0=\beta+1$ and $\tau\circ g_0=\tau-1$, the function $b=\beta+\tau$ is g_0-invariant, so $(\beta2)$ says, by Riemann's theorem, that b as a function of t_c on U_r extends holomorphically across 0. Define a sheaf $\mathfrak{B}$ on B by $\mathfrak{B}(U):=\{\beta\in\mathcal{O}(\widetilde{U}) : (\beta1)\text{ holds}\}$, with the extra requirement, if $p_0\in U$, that $\beta+\tau$ restricted to the distinguished component $U_r(U)$ of Definition 3.9 extend holomorphically at $t_c=0$. The difference of two local sections is a Δ-invariant holomorphic function, i.e. a holomorphic function of t, holomorphic in t_c at p_0; so $\mathfrak{B}$ is a pseudo-torsor under $\mathcal{O}_B$. Local solvability: at $p\in B^\circ$ prescribe $\beta:=0$ on one sheet and extend by $(\beta1)$; at p_0 take $\beta:=-\tau$ on U_r, so $\beta\circ g_0=\beta+1$ and $b=0$; at p_j $(j=1,2)$ set $\beta:=\frac{1}{m_j}\sum_{k=0}^{m_j-1}k\,(\phi_j\circ g_j^k)$ on Δ_j, which by the cocycle relation telescopes to $\beta\circ g_j-\beta=\frac{1}{m_j}\big(m_j\phi_j-\sum_k\phi_j\circ g_j^k\big)=\phi_j$, and extend by equivariance. Hence $\mathfrak{B}$ is an $\mathcal{O}_B$-torsor; $H^1(\mathbb{P}^1,\mathcal{O})=0$ gives a global section and $H^0(\mathbb{P}^1,\mathcal{O})=\mathbb{C}$ gives the affine family $\beta^{\mathrm{part}}+c_0$. ☐

3.4 Nondegeneracy, and the choice of c_0

Lemma 3.14. For every $z\in\mathfrak{h}_z$, viewing $\Pi(z)$ as an $\mathbb{R}$-linear map $\Lambda\otimes\mathbb{R}\to\mathbb{C}^2=\mathbb{R}^4$ in the coordinates $(\operatorname{Re}\zeta_1,\operatorname{Im}\zeta_1,\operatorname{Re}\zeta_2,\operatorname{Im}\zeta_2)$ and the basis $(\hat\gamma,\hat u,\hat w,\hat\delta)$,
$$\det{}_{\mathbb{R}}\Pi(z) = -\det\operatorname{Im}Z(z) = \operatorname{Im}\tau(z)\cdot D(z);$$
thus $L(z)$ is a lattice in $\mathbb{C}^2$ if and only if $D(z)\neq 0$. Moreover D is Δ-invariant.

Proof. For $v_1,v_2\in\mathbb{C}^2$ one has $\det_{\mathbb{R}}(v_1,v_2,e_1,e_2)=-\det\operatorname{Im}[v_1\,v_2]$ (expand along the last two columns; the sign is that of the 4-cycle $(1\,3\,4\,2)$). With $[v_1\,v_2]=Z$ this gives $\det_{\mathbb{R}}\Pi=-\det\operatorname{Im}Z=\operatorname{Im}\tau\operatorname{Im}\beta-6(\operatorname{Im}\mu)^2=\operatorname{Im}\tau\cdot D$.

By §3.5, $\Pi(gz)=R_g(z)\Pi(z)M_g$ with $\det M_g=\det\rho_V(g)=1$, and $\det R_{g_1}=-1/\tau$, $\det R_{g_2}=1/\tau$. Taking $\det_{\mathbb{R}}$ and using $\det_{\mathbb{R}}(R)=|\det_{\mathbb{C}}R|^2$ gives $\operatorname{Im}\tau(gz)D(gz)=|\det R_g|^2\operatorname{Im}\tau(z)D(z)$; for $g=g_1,g_2$ we have $\operatorname{Im}\tau(gz)=\operatorname{Im}\tau/|\tau|^2$ and $|\det R_g|^2=1/|\tau|^2$, whence $D(gz)=D(z)$. As $\Delta=\langle g_1,g_2\rangle$, D is Δ-invariant. ☐

Proposition 3.15. Part (iv) of Theorem 3.4 holds.

Proof. By Lemma 3.14, D descends to a continuous function D_0 on $B \setminus \{p_0\}$, π being an open surjection; in particular D_0 is finite at p_1, p_2, where $\operatorname{Im}\tau = \sqrt{3}/2$ resp. 1 and μ, β are holomorphic. On U_r, $\operatorname{Im}\beta = \operatorname{Im} b - \operatorname{Im}\tau$ with b and μ bounded, so $D \le \operatorname{Im} b - \operatorname{Im}\tau \to -\infty$ as $t_c \to 0$ (Corollary 3.7). Hence there is a compact $K \subset B \setminus \{p_0\}$ with $D_0 < 0$ off K, and $M := \max_K D_0 < \infty$. Replacing β by $\beta + c_0$ replaces D by $D + \operatorname{Im} c_0$, so any c_0 with $\operatorname{Im} c_0 < -\max(M, 0)$ yields (β3): $D < 0$ on all of $\mathfrak{h}_z$, in particular $D(z_1) = \operatorname{Im}\beta(z_1) - 1/\sqrt{3} < 0$ and $D(z_2) = \operatorname{Im}\beta(z_2) - 3/2 < 0$ (using $\operatorname{Im}\mu(z_1) = -\sqrt{3}/6$, $\operatorname{Im}\mu(z_2) = -1/2$). By Lemma 3.14, $L(z)$ is then a lattice for every z, and $\mathcal{T} \to \mathfrak{h}_z$ is a holomorphic family of compact complex 2-tori with holomorphic zero section. □

From now on β is one function satisfying (β1)–(β3); X will depend on this choice only through the constant c_0.

3.5 Equivariance

Identify $\Pi(z)$ with the pair of linear forms $\lambda \mapsto \langle \sigma_i(z), \lambda \rangle$ on Λ, where

$$\sigma_1 := 6\mu\,\gamma + \tau\,u + w, \qquad \sigma_2 := \beta\,\gamma + \mu\,u + \delta \;\in V_{\mathbb{C}};$$

indeed the rows of Π are the values of σ_1, σ_2 on $(\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta})$.

Proposition 3.16. With $F^1(z) := \langle \sigma_1(z), \sigma_2(z) \rangle_{\mathbb{C}} \subset V_{\mathbb{C}}$ one has $F^1(g_j z) = T_j F^1(z)$ and

$$\sigma_1 \circ g_1 = -\tfrac{1}{\tau}\, T_1\sigma_1, \qquad \sigma_2 \circ g_1 = T_1\sigma_2 + \tfrac{1-\mu}{\tau}\, T_1\sigma_1,$$
$$\sigma_1 \circ g_2 = \tfrac{1}{\tau}\, T_2\sigma_1, \qquad \sigma_2 \circ g_2 = T_2\sigma_2 - \tfrac{\mu}{\tau}\, T_2\sigma_1,$$

as well as $\sigma_i \circ g_0 = T_0\sigma_i$. Consequently $R_{g_0} = I$,

$$R_{g_1} = \begin{pmatrix} -1/\tau & 0 \\ (1-\mu)/\tau & 1 \end{pmatrix}, \qquad R_{g_2} = \begin{pmatrix} 1/\tau & 0 \\ -\mu/\tau & 1 \end{pmatrix},$$

and $\Pi(gz) = R_g(z)\Pi(z)M_g$, $R_{gh}(z) = R_g(hz)R_h(z)$ for all $g, h \in \Delta$.

Proof. Direct computation from the matrices of §2 ($T_1\gamma = \gamma$, $T_1 u = -u - w$, $T_1 w = -6\gamma + u$, $T_1\delta = 2\gamma + u + w + \delta$; $T_2 u = 6\gamma + w$, $T_2 w = -u$, $T_2\delta = -3\gamma + u + \delta$; $T_0 = I + N$) gives

$$\begin{aligned}
T_1\sigma_1 &= (6\mu - 6)\gamma + (1-\tau)u - \tau w,\\
T_1\sigma_2 &= (\beta + 2)\gamma + (1-\mu)u + (1-\mu)w + \delta,\\
T_2\sigma_1 &= (6\mu + 6\tau)\gamma - u + \tau w,\\
T_2\sigma_2 &= (\beta + 6\mu - 3)\gamma + u + \mu w + \delta,\\
T_0\sigma_1 &= 6\mu\gamma + (\tau - 1)u + w,\\
T_0\sigma_2 &= (\beta + 1)\gamma + \mu u + \delta.
\end{aligned}$$

Dividing $T_1\sigma_1$ by $-\tau$ normalises the w-coefficient to 1 and leaves u-coefficient $(\tau - 1)/\tau = \tau \circ g_1$ and γ-coefficient $6(1-\mu)/\tau = 6(\mu \circ g_1)$ by (τ2), (μ1); so the result is $\sigma_1(g_1 z)$. Subtracting $(1-\mu)$ times it from $T_1\sigma_2$ kills the w-coefficient and leaves u-coefficient $(1-\mu)(1 - \frac{\tau-1}{\tau}) = \frac{1-\mu}{\tau} = \mu \circ g_1$ and γ-coefficient $\beta + 2 - 6(1-\mu)^2/\tau = \beta \circ g_1$ by (β1), i.e. $\sigma_2(g_1 z)$. The same normalisation applied to the T_2-row gives $\tau \circ g_2 = -1/\tau$, $\mu \circ g_2 = 1 + \mu/\tau$, $\beta \circ g_2 = \beta - 3 - 6\mu^2/\tau$, and the T_0-row is already $(\sigma_1(g_0 z), \sigma_2(g_0 z))$, so $R_{g_0} = I$.

If $\sigma_i(gz) = \sum_k (R_g)_{ik}\rho_V(g)\sigma_k(z)$ then, for $\lambda \in \Lambda$, $\langle \sigma_i(gz), \lambda \rangle = \sum_k (R_g)_{ik} \langle \sigma_k(z), \rho_V(g)^t\lambda \rangle$, i.e. $\Pi(gz) = R_g(z)\Pi(z)M_g$; equivalently $R_g(z)\Pi(z) = \Pi(gz)M_g^{-1}$. Since the right $(\hat{w}, \hat{\delta})$-block of Π is I, the right block of $\Pi(gz)M_g^{-1} = R_g\Pi(z)$ is R_g, the matrix so called in Definition 3.3; it is invertible because $R_g\Pi(z) = \Pi(gz)M_g^{-1}$ has rank 2. Finally $M_{gh} = \rho_V(gh)^t = M_h M_g$, so

$$\Pi(ghz) = R_g(hz)R_h(z)\Pi(z)M_h M_g = R_g(hz)R_h(z)\Pi(z)M_{gh},$$

and the cocycle relation follows from the uniqueness of R_{gh}. □

3.6 The family $J \to B^\circ$ and its marking

Convention 3.17. Paths are composed left to right: if $c(1) = c'(0)$ then cc' means "first c, then c'". With these conventions the map of §2 sending the homotopy class of $\pi \circ c$, for a path c from the base point z to gz in $\mathfrak{h}_z^\circ$, to $g \in \Delta$ is a group homomorphism $\pi_1(B^\circ, \pi(z)) \to \Delta$.

Orientations of meridians. Since $s_j \circ g_j = e^{-2\pi i/m_j} s_j$ and $t_j \circ \pi = s_j^{m_j}$, the deck transformation g_j rotates the linearising coordinate *clockwise*, so it is the *clockwise* meridian at p_j (and likewise the clockwise loop at the cusp p_0) that lifts to g_j (resp. g_0); the counterclockwise meridians lift to g_j^{-1}, resp. g_0^{-1}. We therefore declare: *all meridians used for the marking are oriented clockwise*, i.e. negatively for the complex orientation of B, and then the product of the meridians around p_1, p_2, p_0, taken in this order and suitably based, is trivial, matching $g_1 g_2 g_0 = 1$ in Δ. Equivalently, if one insists on counterclockwise meridians $\check{\sigma}_1, \check{\sigma}_2, \check{\sigma}_0$, these lift to $g_1^{-1}, g_2^{-1}, g_0^{-1}$ and the relation $g_1 g_2 g_0 = 1$ reads $\check{\sigma}_0 \check{\sigma}_2 \check{\sigma}_1 = 1$ in $\pi_1(B^\circ)$, i.e. the product of the three counterclockwise meridians taken in the *opposite* order.

Remark 3.18. Reversing the orientations of all three meridians simultaneously replaces the marking by its inverse and hence replaces p by $-p$ in §7. All statements there involve only $|p|$, so they hold equally for either orientation, and in particular for the choice made in Convention 3.17.

Proposition 3.19. Part (v) of Theorem 3.4 holds.

Proof. By Proposition 3.16, $\tilde{g}(z, \zeta) = (gz, R_g(z)\zeta)$ satisfies $\tilde{g} \circ \tilde{h} = \widetilde{gh}$, so $g \mapsto \tilde{g}$ is an action of Δ on $\mathfrak{h}_z \times \mathbb{C}^2$ by biholomorphisms over the Δ-action on $\mathfrak{h}_z$. It normalises the Λ-action, since $R_g(z)\Pi(z) = \Pi(gz)M_g^{-1}$ gives

$$\tilde{g}\big(z, \zeta + \Pi(z)\lambda\big) = \big(gz,\ R_g(z)\zeta + \Pi(gz)M_g^{-1}\lambda\big),$$

so it descends to $\mathcal{T}$ and maps $L(z)$-cosets to $L(gz)$-cosets. By §2, Δ acts on $\mathfrak{h}_z$ properly discontinuously with stabilisers trivial outside $\Delta z_1 \cup \Delta z_2$ and cyclic of orders 3, 4 at z_1, z_2; hence it acts freely and properly discontinuously on $\mathfrak{h}_z^\circ$, and therefore on $\mathcal{T}|_{\mathfrak{h}_z^\circ}$. So J is a complex 3-manifold; the induced map to B° is holomorphic, proper with fibres the compact tori $\mathbb{C}^2/L(z)$, and a submersion, and the zero section descends because $R_g(z)0 = 0$.

Marking and monodromy. We use Convention 3.17. For $z \in \mathfrak{h}_z^\circ$ identify H_1 of the fibre $\mathbb{C}^2/L(z)$ with Λ by $\lambda \mapsto [\,0 \to \Pi(z)\lambda\,]$. Let c be a path from z to gz in $\mathfrak{h}_z^\circ$, so that $\pi \circ c$ is a loop in B°; the fibre of J over its base point is identified with the fibre of $\mathcal{T}$ at z, and with the fibre at gz through $\tilde{g}$. Parallel transport in $\mathcal{T}$ along c is tautological, i.e. the identity in the Λ-coordinates, and by

$$R_g(z)\Pi(z)\lambda = \Pi(gz)M_g^{-1}\lambda$$

the map $\tilde{g}$ carries the cycle with coordinate λ at z to the cycle with coordinate $M_g^{-1}\lambda$ at gz. Hence the monodromy on H_1, read in the marking at the base point, is $\lambda \mapsto M_g^{-1}\lambda$.

We now fix the meaning of "monodromy" used here and in §7. The two identifications of the fibre of J over the base point used above — with $\mathcal{T}_z$ at the start of the loop and with $\mathcal{T}_{gz}$ at its end — differ by $\tilde{g}$, and $\tilde{g}_* = M_g^{-1}$ in the Λ-coordinates; consequently *forward* parallel transport of H_1 around the loop lifting to g, read in the marking at the base point, is $\tilde{g}_*^{-1}$, i.e. $\lambda \mapsto M_g\lambda$, while the *monodromy representation* is by definition the inverse of forward transport,

$$g \longmapsto M_g^{-1}.$$

This is the convention (the inverse-transport, or "representation", convention of §7) in which the assignment is a homomorphism: $M_{gh}^{-1} = M_g^{-1}M_h^{-1}$ since $M_{gh} = M_h M_g$, whereas forward transport $g \mapsto M_g$ is anti-multiplicative for the left-to-right composition of Convention 3.17. The two conventions differ by inverting all the matrices, which changes neither the group $\langle A_1, A_2 \rangle$ nor the invariant $|p|$ of §7 (cf. Remark 3.18), so the statements of §7 hold in either one.

For the clockwise meridians of p_1, p_2, p_0 — i.e. the meridians of Convention 3.17, which lift to g_1, g_2, g_0 — the matrices of the monodromy representation are A_1, A_2, M_0, and $A_1 A_2 M_0 = I$ matches $g_1 g_2 g_0 = 1$.

Finally, for $j = 1, 2$ the fibre of $\mathcal{T}$ over z_j is $A_0^{(j)} = \mathbb{C}^2/L(z_j)$, a torus by Proposition 3.15, and $\tilde{g}_j$ restricts to the automorphism $\zeta \mapsto R_{g_j}(z_j)\zeta$ of order m_j. ☐

Remark 3.20. In the flat real trivialisation $\mathcal{T} \cong \mathfrak{h}_z \times (\Lambda \otimes \mathbb{R})/\Lambda$, $(z, x) \mapsto (z, \Pi(z)x)$, the map $\tilde{g}$ reads $(z, x) \mapsto (gz, M_g^{-1}x)$, i.e. $(z, x) \mapsto (g_j z, A_j x)$ for $g = g_j$. This is the form used in §5.

3.7 The local form of the periods at p_0

Recall $s = \tau - h(t_c \circ \pi)$ from Corollary 3.7, and that $B_0 \colon \overline{\Lambda} \to \Lambda_{\mathrm{tor}}$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the coordinates of §2.

Proposition 3.21. On U_r put $b := \beta + \tau$. Then μ and b are g_0-invariant and define holomorphic functions $\mu(t_c), b(t_c)$ on $\{|t_c| < r\}$, h is holomorphic there, and with

$$C(t_c) := \begin{pmatrix} 6\mu(t_c) & h(t_c) \\ b(t_c) - h(t_c) & \mu(t_c) \end{pmatrix}$$

one has $Z(z) = s(z)B_0 + C(t_c \circ \pi(z))$ for $z \in U_r$; in particular C is holomorphic *at* $t_c = 0$. Moreover, for $\bar{\lambda} \in \overline{\Lambda} = \mathbb{Z}^2$,

$$\exp\bigl(2\pi i\, Z\bar{\lambda}\bigr) = t_c^{B_0\bar{\lambda}} \cdot c_{\bar{\lambda}}(t_c), \qquad c_{\bar{\lambda}}(t_c) = \exp\bigl(2\pi i\, C(t_c)\bar{\lambda}\bigr),$$

the exponentials being componentwise and $t_c^{(c,d)} := (t_c^c, t_c^d)$.

Proof. $\mu \circ g_0 = \mu$ and $b \circ g_0 = b$ by $(\mu 1)$, $(\beta 1)$, $(\tau 2)$; holomorphy at $t_c = 0$ is Theorem 3.4(ii),(iii) (where the regularity is asserted precisely on the distinguished component U_r), and h is holomorphic on $\{|t_c| < r\}$ because u_1 is holomorphic and nowhere zero there (Proposition 3.6). Now $\tau = s + h$ and $\beta = b - s - h$, so

$$Z = \begin{pmatrix} 6\mu & \tau \\ \beta & \mu \end{pmatrix} = \begin{pmatrix} 6\mu & s + h \\ -s + b - h & \mu \end{pmatrix} = s\, B_0 + C(t_c).$$

Finally $\exp(2\pi i\, sB_0\bar{\lambda}) = (e^{2\pi i s})^{B_0\bar{\lambda}} = t_c^{B_0\bar{\lambda}}$, since $B_0\bar{\lambda} \in \mathbb{Z}^2$ and $e^{2\pi i s} = t_c \circ \pi$. ☐

Remark 3.22. The regularity of C at $t_c = 0$ comes from the two torsor problems of §§3.2,3.3: their sheaves $\mathfrak{L}$ and $\mathcal{O}_B$ are the sheaves of local solutions *with* the prescribed regularity at the distinguished cusp over p_0, and they still have local sections there ($\mu = 0$, resp. $\beta = -\tau$). It is thus obtained independently of the nilpotent orbit theorem, which does not apply in the present setting (Remark 3.23).

3.8 The Hodge form: signature $(1, 1)$

Remark 3.23. Let Q_0 be the $\langle T_1, T_2 \rangle$-invariant alternating form of §2. Then $Q_0(\sigma_1, \sigma_2) = 6\mu\, Q_0(\gamma, \delta) + \mu\, Q_0(w, u) = 6\mu - 6\mu = 0$, so $F^1(z)$ is Q_0-Lagrangian, and the Hermitian form $\mathsf{h}(v, v') := iQ_0(v, \overline{v'})$ on $F^1(z)$ has Gram matrix

$$-\begin{pmatrix} 12\,\mathrm{Im}\,\tau & 12\,\mathrm{Im}\,\mu \\ 12\,\mathrm{Im}\,\mu & 2\,\mathrm{Im}\,\beta \end{pmatrix}, \qquad \det = 24\,\mathrm{Im}\,\tau \cdot D < 0.$$

Thus $\mathsf{h}|_{F^1}$ is nondegenerate of signature $(1, 1)$ for either sign of Q_0: the fibres of J carry no monodromy-compatible polarisation, and the arguments of this paper use none. The conclusion holds for every choice of c_0, and indeed for every T_0-equivariant family on a punctured neighbourhood of p_0: positivity with respect to $\pm Q_0$ would force $\mathrm{Im}\,\tau$ and $\mathrm{Im}\,\beta$ to have the same strict sign, while both $e^{\pm 2\pi i\tau}$, $e^{\pm 2\pi i\beta}$ (signs making them bounded) are g_0-invariant, hence holomorphic in t_c and vanishing at 0 to orders $\nu, \nu' \geq 1$ (a non-zero value would make τ, resp. β, single-valued), so that continuation along the same loop would shift τ and β by $-\nu$ and $-\nu'$, of the *same* sign; but $(\tau 2)$, $(\beta 1)$ give the shifts -1 and $+1$. Invariantly, the form $b_N(x, y) := Q_0(x, Ny)$ on $V/\ker N = \langle \bar{w}, \bar{\delta} \rangle$ is $\mathrm{diag}(6, -1)$, indefinite, so (V, Q_0, N) supports no polarised limit mixed Hodge structure.

4 The filling at p_0: a toric quotient.

4.1 Conventions and input from §3

Throughout §4 we use the lattice data of §2 and the period functions of §3. Positive radii are denoted $\epsilon, \epsilon_0, \epsilon_1, r, \rho$, and are thus distinct from the lattice vectors $\varepsilon = \hat{\gamma} + 2\hat{u} - 4\hat{w}$, $\varepsilon' = \hat{\gamma} + 3\hat{u} - 3\hat{w} \in \Lambda$; the toric prime divisors of the model below are written E_v ($v \in \mathbb{Z}^2$), so that D remains available for the discs $D_j \subset B$ and for $D = \operatorname{Im}\beta - 6(\operatorname{Im}\mu)^2/\operatorname{Im}\tau$. The three consequences of Theorem 3.4 recorded next are the principal ones needed below; §4.7 uses in addition the cocycle relation $\Pi(gz) = R_g(z)\Pi(z)M_g$ of Proposition 3.16 and the description of J over $\{0 < |t_c| < \epsilon\}$ as $(U_\epsilon \times \mathbb{C}^2)/G$ from Proposition 2.11(iii).

(P1) There is $r > 0$ such that $s = \tau - h(t_c \circ \pi)$ maps the cusp neighbourhood $U_r \subset \mathfrak{h}_z$ biholomorphically onto $\{\operatorname{Im} s > \frac{1}{2\pi}\log(1/r)\}$, with $e^{2\pi i s} = t_c \circ \pi$ and $s \circ g_0 = s - 1$.

(P2) On U_r, $Z(z) = s(z)\, B_0 + C(t_c \circ \pi(z))$ with C holomorphic on the full disc $\{|t_c| < r\}$; recall $c_{\bar{\lambda}}(t_c) = \exp\big(2\pi i\, C(t_c)\bar{\lambda}\big) \in (\mathbb{C}^*)^2$ for $\bar{\lambda} \in \overline{\Lambda} \cong \mathbb{Z}^2$.

(P3) $\Pi(g_0 z)M_{g_0}^{-1} = [\, Z(z) \mid I\,] = \Pi(z)$; in particular $R_{g_0} \equiv I$, i.e. $\tilde{g}_0(z,\zeta) = (g_0 z, \zeta)$.

Put $\mathsf{R}(t_c) := -2\pi \operatorname{Im} C(t_c) \in M_2(\mathbb{R})$ and $\|\mathsf{R}\|_\infty := \sup_{|t_c| \le r/2} \|\mathsf{R}(t_c)\| < \infty$ (operator norm; the supremum is finite by (P2)). Since $B_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is orthogonal,

$$|B_0\bar{\lambda}| = |\bar{\lambda}| \quad (\bar{\lambda} \in \mathbb{R}^2), \qquad \|B_0\| = 1, \qquad |\det B_0| = 1. \tag{4.1}$$

The data B_0 and C will enter only through (4.1) and (P2); see Remark 4.9.

4.2 The toric model $Y = Y_{\mathfrak{F}}$

Notation 4.1. $N' := \mathbb{Z}^3$ with standard basis e_1, e_2, e_3, points written (y, y_3), $y \in \mathbb{R}^2$; $M' := \operatorname{Hom}(N', \mathbb{Z})$, with characters χ^m of $T_{N'} = N' \otimes \mathbb{C}^* = (\mathbb{C}^*)^3$ and coordinates $x_1 = \chi^{(1,0,0)}$, $x_2 = \chi^{(0,1,0)}$, $t = \chi^{(0,0,1)}$. The cocharacter lattice $\mathbb{Z}^2$ of $\{(x_1, x_2, 1)\}$ is identified with Λ_{tor} by $e_1 \leftrightarrow \hat{w}$, $e_2 \leftrightarrow \hat{\delta}$; then $B_0 : \overline{\Lambda} \to \Lambda_{\mathrm{tor}} = \mathbb{Z}^2$, $B_0\bar{\hat{\gamma}} = -\hat{\delta}$, $B_0\bar{\hat{u}} = \hat{w}$.

$\mathcal{T}_{A_2}$ is the triangulation of $\mathbb{R}^2$ with vertex set $\mathbb{Z}^2$ and triangles $\{v, v+e_1, v+e_2\}$, $\{v+e_1, v+e_2, v+e_1+e_2\}$ ($v \in \mathbb{Z}^2$); equivalently it is cut out by the three pencils $y_1 \in \mathbb{Z}$, $y_2 \in \mathbb{Z}$, $y_1 + y_2 \in \mathbb{Z}$. Its edge directions are $\pm e_1, \pm e_2, \pm(e_1 - e_2)$, and it is invariant under all integral translations. For a cell (vertex, edge, triangle) P put $\hat{P} := \operatorname{cone}(P \times \{1\}) \subset N'_{\mathbb{R}}$ and

$$\mathfrak{F} := \{0\} \cup \{\hat{P} : \ P \text{ a cell of } \mathcal{T}_{A_2}\}, \qquad Y := Y_{\mathfrak{F}}.$$

For a triangle σ, $\mathcal{U}_\sigma := \operatorname{Spec} \mathbb{C}[\hat{\sigma}^\vee \cap M']$; and $Y_\epsilon := \{p \in Y : \ |t(p)| < \epsilon\}$.

Lemma 4.2. (i) $\mathfrak{F}$ is a fan of strongly convex rational polyhedral cones with $|\mathfrak{F}| = (\mathbb{R}^2 \times \mathbb{R}_{>0}) \cup \{0\}$, and every cone of $\mathfrak{F}$ is unimodular. Consequently Y is a connected, Hausdorff, second countable complex 3-manifold containing $T_{N'}$ as a dense open subset, with holomorphic $T_{N'}$-action.

(ii) t is holomorphic on Y with $t^{-1}(\mathbb{C}^*) = T_{N'}$ and $t^{-1}(0) = \bigcup_{v \in \mathbb{Z}^2} E_v$, where E_v is the divisor of the ray $\hat{v} = \mathbb{R}_{\ge 0}(v, 1)$. Moreover all these rays lie at height 1, i.e. $\langle (0,0,1), (v,1) \rangle = 1$ for every $v \in \mathbb{Z}^2$; hence the divisor of t on Y is

$$\operatorname{div}(t) = \sum_{v \in \mathbb{Z}^2} E_v,$$

with all multiplicities equal to 1, and the scheme-theoretic fibre $t^{-1}(0)$ is *reduced* (a normal crossings divisor, by (iii)).

(iii) If σ has vertices v_0, v_1, v_2 and $m_0, m_1, m_2 \in M'$ is the basis dual to $(v_0, 1), (v_1, 1), (v_2, 1)$, then $(z_0, z_1, z_2) := (\chi^{m_0}, \chi^{m_1}, \chi^{m_2})$ is a global coordinate system on $\mathcal{U}_\sigma \cong \mathbb{C}^3$ and $t = z_0 z_1 z_2$ on $\mathcal{U}_\sigma$; moreover $E_{v_i} \cap \mathcal{U}_\sigma = \{z_i = 0\}$.

(iv) $E_v \cong \mathrm{dP}_6$, the smooth toric del Pezzo surface of degree 6; its toric boundary is the anticanonical cycle of six (-1)-curves $C^{(v)}_{v'} := E_v \cap E_{v'}$, $v' - v \in \{\pm e_1, \pm e_2, \pm(e_1 - e_2)\}$, and $E_v \cap E_{v'} = \emptyset$ for all other $v' \neq v$. Three divisors $E_{v_0}, E_{v_1}, E_{v_2}$ meet (in one point) iff $\{v_0, v_1, v_2\}$ is a triangle of $\mathcal{T}_{A_2}$.

Proof. (i) A direct computation gives $\det\big((v_0,1),(v_1,1),(v_2,1)\big) = \det(e_1,e_2) = 1$ resp. $\det(e_2 - e_1, e_2) = -1$ for the two types of triangle, so every $\hat\sigma$ and hence every face is unimodular. For $y_3 > 0$, $(y, y_3) \in \hat P$ iff $y/y_3 \in P$, and $\hat P \cap (\mathbb{R}^2 \times \{0\}) = \{0\}$ as P is bounded; hence $\hat P \cap \hat P' = \widehat{P \cap P'}$, a common face, and $\mathfrak{F}$ is a fan with the stated support. Smoothness, separatedness and the orbit–cone correspondence for $Y = \bigcup_\sigma \mathcal{U}_\sigma$ are the standard toric statements [Ful93, §1.4, §3.1], [Oda88, §1.2–1.3]; they concern single cones and pairs of cones, hence hold verbatim for the infinite fan $\mathfrak{F}$ (only quasi-compactness is lost), as in [Mum72], [KKMS], [AMRT]. In particular, for a triangle σ the orbits contained in $\mathcal{U}_\sigma$ are exactly the $O_{\hat P}$ with $\hat P$ a face of $\hat\sigma$, so the only divisors E_v meeting $\mathcal{U}_\sigma$ are those with v a vertex of σ. $\mathfrak{F}$ is countable, whence second countability; all charts meet the dense irreducible $T_{N'}$, whence connectedness.

(ii) $(0,0,1)$ takes the value $y_3 \geq 0$ on $|\mathfrak{F}|$, so t is regular on Y, and invertible on $O_{\hat P}$ iff $(0,0,1) \in \hat P^\perp$, i.e. iff $\hat P = 0$; so $t^{-1}(0)$ is the union of the orbit closures of the rays. The order of vanishing of $\chi^{(0,0,1)}$ along E_v is $\langle (0,0,1), (v,1)\rangle = 1$, since every primitive generator of a ray of $\mathfrak{F}$ has height 1; this gives $\mathrm{div}(t) = \sum_v E_v$ with all multiplicities 1. Reducedness of the scheme $t^{-1}(0)$ is then local and immediate from (iii): in the chart $\mathcal{U}_\sigma \cong \mathbb{C}^3$ one has $t = z_0 z_1 z_2$, and $\mathbb{C}[z_0, z_1, z_2]/(z_0 z_1 z_2)$ is reduced. (Note that this is a statement *upstairs*, on Y, proved before and independently of any quotient construction; it will be quoted in Theorem 4.5(d) and in Proposition 4.6(i).)

(iii) Unimodularity gives $\hat\sigma^\vee \cap M' = \sum_i \mathbb{Z}_{\geq 0} m_i$ with (m_i) a basis of M', so $\mathcal{U}_\sigma \cong \mathbb{C}^3$; and $\langle (0,0,1), (v_j, 1)\rangle = 1$ gives $(0,0,1) = m_0 + m_1 + m_2$, i.e. $t = z_0 z_1 z_2$. The last assertion is the orbit–cone correspondence: $E_{v_i} \cap \mathcal{U}_\sigma$ is the closure of $O_{\hat v_i}$ in $\mathcal{U}_\sigma$, cut out by the characters χ^m with $m \in \hat\sigma^\vee$, $\langle m, (v_i, 1)\rangle > 0$, i.e. by z_i.

(iv) The fan of E_v is the star of $\hat v$ in $N'/\mathbb{Z}(v,1)$; translating by $-v$ (Lemma 4.3(i)) take $v = 0$, so $N'/\mathbb{Z}e_3 \cong \mathbb{Z}^2$ by $(y, y_3) \mapsto y$. The six triangles containing 0 are the cones over consecutive pairs of

$$e_1, \quad e_2, \quad e_2 - e_1, \quad -e_1, \quad -e_2, \quad e_1 - e_2,$$

each consecutive pair of determinant $+1$: the complete smooth hexagonal fan of dP_6. The invariant curves are the six (-1)-curves of the anticanonical hexagon, that of the ray through v' being $E_0 \cap E_{v'}$; the rest is the orbit–cone correspondence. □

4.3 The automorphisms $\Psi_{\bar\lambda}$

Lemma 4.3. For $\bar\lambda \in \overline{\Lambda} \cong \mathbb{Z}^2$ let $\varphi_{\bar\lambda} \in GL(N')$ be $\varphi_{\bar\lambda}(y, y_3) = (y + y_3 B_0 \bar\lambda,\ y_3)$.

(i) $\varphi_{\bar\lambda}$ maps $\hat P$ onto $\widehat{P + B_0\bar\lambda}$ for every cell P; in particular $\varphi_{\bar\lambda}(\mathfrak{F}) = \mathfrak{F}$, and $\bar\lambda \mapsto \varphi_{\bar\lambda}$ is a homomorphism. As $|\det B_0| = 1$, $B_0 \overline{\Lambda} = \mathbb{Z}^2$ and $\{\varphi_{\bar\lambda}\}$ induces on $y_3 = 1$ the group of *all* translations by $\mathbb{Z}^2 = \Lambda_{\mathrm{tor}}$.

(ii) The induced toric automorphism $\Phi_{\bar\lambda}$ of Y satisfies $\Phi_{\bar\lambda}(\mathcal{U}_\sigma) = \mathcal{U}_{\sigma + B_0\bar\lambda}$, $\Phi_{\bar\lambda}(O_{\hat P}) = O_{\widehat{P + B_0\bar\lambda}}$, $t \circ \Phi_{\bar\lambda} = t$, and $\Phi_{\bar\lambda}(x, t) = (t^{B_0\bar\lambda} x, t)$ on the torus.

(iii) For $0 < \epsilon \leq r$ the map $\Psi_{\bar\lambda}(p) := (c_{\bar\lambda}(t(p)), 1) \cdot \Phi_{\bar\lambda}(p)$ is a biholomorphism of Y_ϵ with $t \circ \Psi_{\bar\lambda} = t$, and $\bar\lambda \mapsto \Psi_{\bar\lambda}$ is an action of $\overline{\Lambda} \cong \mathbb{Z}^2$ on Y_ϵ by biholomorphisms. On the torus,

$$\Psi_{\bar\lambda}(x, t) = \big(c_{\bar\lambda}(t)\, t^{B_0\bar\lambda}\, x,\ t\big). \tag{4.2}$$

Proof. (i) $\varphi_{\bar\lambda}$ is integral, preserves y_3, and on $y_3 = 1$ is the translation by $B_0\bar\lambda \in \mathbb{Z}^2$, under which $\mathcal{T}_{A_2}$ is invariant; $\varphi_{\bar\lambda}\varphi_{\bar\mu} = \varphi_{\bar\lambda + \bar\mu}$ is clear, and $B_0\overline{\Lambda} = \mathbb{Z}^2$ since $|\det B_0| = 1$.

(ii) Only the torus formula needs comment: $\varphi_{\bar\lambda} \otimes \mathbb{C}^*$ sends $\sum_i e_i \otimes x_i + e_3 \otimes t$ to $\sum_i e_i \otimes x_i + (e_3 + B_0\bar\lambda) \otimes t = (t^{B_0\bar\lambda} x, t)$, and $\varphi^*_{\bar\lambda}(0,0,1) = (0,0,1)$ gives $t \circ \Phi_{\bar\lambda} = t$.

(iii) As t is holomorphic on Y and C on $\{|t_c| < r\}$, the map $\tau_{\bar\lambda}(p) := (c_{\bar\lambda}(t(p)), 1) \cdot p$ is holomorphic on Y_ϵ with holomorphic inverse $p \mapsto (c_{\bar\lambda}(t(p))^{-1}, 1) \cdot p$; the acting element has t-coordinate 1, so $t \circ \tau_{\bar\lambda} = t$ and

$\tau_{\bar\lambda}$ preserves every $T_{N'}$-orbit. Hence $\Psi_{\bar\lambda} = \tau_{\bar\lambda} \circ \Phi_{\bar\lambda}$ is a biholomorphism of Y_ϵ preserving t. Since $\varphi_{\bar\lambda}$ fixes e_1, e_2 and $t \circ \Phi_{\bar\lambda} = t$, we get $\Phi_{\bar\lambda}\tau_{\bar\mu} = \tau_{\bar\mu}\Phi_{\bar\lambda}$; and $c_{\bar\lambda}c_{\bar\mu} = c_{\bar\lambda+\bar\mu}$ by linearity of $C(t)\bar\lambda$ in $\bar\lambda$, so $\Psi_{\bar\lambda}\Psi_{\bar\mu} = \Psi_{\bar\lambda+\bar\mu}$. Formula (4.2) is (ii). □

4.4 Charts and the rescaled position map

For $p = (x, t) \in T_{N'}$ with $0 < |t| < 1$ set

$$\operatorname{Log} p := (\log|x_1|, \log|x_2|, \log|t|), \qquad y(p) := \frac{\log|x(p)|}{\log|t(p)|} \in \mathbb{R}^2,$$

so $\operatorname{Log} p = \log|t| \cdot (y(p), 1)$ and $|\chi^m(p)| = e^{\langle m, \operatorname{Log} p\rangle}$. For a triangle $\sigma = \{v_0, v_1, v_2\}$ let θ_i^σ be the barycentric coordinates, $(y, 1) = \sum_i \theta_i^\sigma(y)(v_i, 1)$, and for $\eta \ge 0$

$$\sigma^{(\eta)} := \{y \in \mathbb{R}^2 : \ \theta_i^\sigma(y) > -\eta \ (i = 0, 1, 2)\},$$

the dilation of the open triangle about its barycentre by $1 + 3\eta$; it is bounded of diameter $\le (1 + 3\eta)\sqrt{2}$. With $z_i = \chi^{m_i}$ as in Lemma 4.2(iii),

$$\log|z_i(p)| = \langle m_i, \operatorname{Log} p\rangle = \log|t(p)| \cdot \theta_i^\sigma(y(p)). \tag{4.3}$$

Write $\mathcal{U}_\sigma(S) := \{p \in \mathcal{U}_\sigma : |z_i(p)| < S \ \forall i\}$, $\mathcal{U}_\sigma(S, \epsilon) := \mathcal{U}_\sigma(S) \cap Y_\epsilon$, and $\bar{\mathcal{U}}_\sigma(1) := \{|z_i| \le 1 \ \forall i\}$, a compact subset of $\mathcal{U}_\sigma \cap \{|t| \le 1\}$.

Lemma 4.4. Let $S \ge 1$, $0 < \epsilon < 1$.

(i) For a torus point p with $|t(p)| < \epsilon$,

$$p \in \mathcal{U}_\sigma(S) \iff y(p) \in \sigma^{(\eta(p))}, \qquad \eta(p) := \frac{\log S}{|\log|t(p)||} \le \eta_0 := \frac{\log S}{|\log \epsilon|};$$

in particular $p \in \mathcal{U}_\sigma(S) \Rightarrow y(p) \in \sigma^{(\eta_0)}$, and $p \in \bar{\mathcal{U}}_\sigma(1) \iff y(p) \in \sigma$.

(ii) Every point of $\{|t| < 1\} \subset Y$ lies in $\bar{\mathcal{U}}_\sigma(1)$ for some triangle σ; more precisely, if $p \in O_{\hat P}$ for a cell P of $\mathcal{T}_{A_2}$, then σ may be chosen with $\sigma \supseteq P$, so that in particular every vertex of P is a vertex of σ. Consequently $\{\mathcal{U}_\sigma(2, \epsilon)\}_\sigma$ is an open cover of Y_ϵ for $\epsilon \le 1$, and every compact subset of Y_ϵ is covered by finitely many $\mathcal{U}_\sigma(2, \epsilon)$.

Proof. (i) By (4.3) and $\log|t| < 0$, $|z_i(p)| < S \iff \theta_i^\sigma > \log S / \log|t| = -\eta(p)$; the last statement is $S = 1$ with non-strict inequalities.

(ii) For a torus point, i.e. for $\hat P = 0$, choose $\sigma \ni y(p)$ and apply (i); every triangle then trivially contains the empty cell. Let now $p \in O_{\hat P}$ with P a vertex, edge or triangle, and let $\sigma \supseteq P$ be a triangle, so $p \in \mathcal{U}_\sigma$; we show that $\sigma \supseteq P$ may be chosen with $p \in \bar{\mathcal{U}}_\sigma(1)$, which is the refined assertion. In the coordinates of Lemma 4.2(iii), $z_i(p) = 0$ for $v_i \in P$, while for $v_i \notin P$ we have $m_i \in \hat P^\perp$, so $z_i|_{O_{\hat P}}$ is the character $\bar m_i$ of $O_{\hat P} = T_{N'(\hat P)}$, $N'(\hat P) := N'/(N' \cap \operatorname{span} \hat P)$. The classes $\overline{(v_i, 1)}$, $v_i \notin P$, generate the image cone $\bar\sigma$, with dual basis $\{\bar m_i\}$; hence

$$p \in \bar{\mathcal{U}}_\sigma(1) \iff \langle \bar m_i, \operatorname{Log}_{\hat P} p\rangle \le 0 \ \ \forall v_i \notin P \iff -\operatorname{Log}_{\hat P}(p) \in \bar\sigma,$$

and it suffices that the cones $\bar\sigma$, $\sigma \supseteq P$, cover $N'(\hat P)_\mathbb{R}$. Let $n \in N'(\hat P)_\mathbb{R}$, lift to $\tilde n \in N'_\mathbb{R}$, and let p_0 be a relative interior point of $P \times \{1\}$. For small $u > 0$ the rescaling of $p_0 + u\tilde n$ to height 1 lies in a triangle of $\mathcal{T}_{A_2}$, and by local finiteness only triangles containing P meet a small neighbourhood of p_0; fixing such u and $\sigma \supseteq P$ we get $p_0 + u\tilde n \in \hat\sigma$, and since $p_0 \in \operatorname{span} \hat P$ has class 0, $n = \overline{(p_0 + u\tilde n)}/u \in \bar\sigma$. The last statements follow from $\bar{\mathcal{U}}_\sigma(1) \subset \mathcal{U}_\sigma(2)$. □

4.5 The filling N_0

Theorem 4.5. Let ϵ_0 satisfy $0 < \epsilon_0 \le r/2$, $\epsilon_0 < 1$ and $|\log \epsilon_0| \ge 2\|\mathrm{R}\|_\infty$, and let $0 < \epsilon \le \epsilon_0$. Then:

(a) the action $\bar\lambda \mapsto \Psi_{\bar\lambda}$ of $\overline{\Lambda} \cong \mathbb{Z}^2$ on Y_ϵ is free and properly discontinuous, i.e. $\{\bar\lambda : \Psi_{\bar\lambda} K \cap K \ne \emptyset\}$ is finite for every compact $K \subset Y_\epsilon$;

(b) $N_0 := Y_\epsilon/\overline{\Lambda}$ is a connected Hausdorff complex 3-manifold, $\mathrm{pr} : Y_\epsilon \to N_0$ is a holomorphic covering, and t descends to a holomorphic $f_0 : N_0 \to \Delta_\epsilon := \{|t_c| < \epsilon\}$;

(c) f_0 is proper and surjective, a submersion over $\Delta_\epsilon \setminus \{0\}$, and for $0 < |t_0| < \epsilon$

$$\begin{aligned} f_0^{-1}(t_0) &= \big((\mathbb{C}^*)^2 \times \{t_0\}\big)/\{c_{\bar\lambda}(t_0)\, t_0^{B_0\bar\lambda}\}_{\bar\lambda\in\overline{\Lambda}} \\ &= \mathbb{C}^2/\big(\mathbb{Z}^2 \oplus Z(s)\mathbb{Z}^2\big) \;=\; \mathbb{C}^2/L(z), \end{aligned}$$

a compact complex 2-torus, where $e^{2\pi i s} = t_0$ and $z \in \mathcal{U}_\epsilon$ is any point with $s(z) = s$;

(d) $W := f_0^{-1}(0)$ is a reduced, connected, compact complex surface and $f_0^*(0) = W$ as divisors — reducedness being inherited from the reducedness of $t^{-1}(0)$ upstairs, Lemma 4.2(ii), since pr is a local biholomorphism carrying t to f_0; more precisely, every $p \in W$ admits local holomorphic coordinates (z_0, z_1, z_2) centred at p in which

$$f_0 = \begin{cases} z_0, & p \text{ on one local branch of } W \text{ (a smooth point of } W), \\ z_0 z_1, & p \text{ on two local branches (a double point)}, \\ z_0 z_1 z_2, & p \text{ on three local branches (a triple point)}, \end{cases}$$

according as p lies on one, two or three of the divisors E_v upstairs; in particular W is a reduced normal crossings surface, smooth away from the double curves $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ and with exactly the two triple points P, Q of Proposition 4.6;

(e) K_{N_0} is trivial: $\Omega := \dfrac{dx_1 \wedge dx_2 \wedge dt}{x_1 x_2}$ extends to a nowhere vanishing holomorphic 3-form on Y and descends to N_0.

Proof. Step 0. By Lemma 4.3, $\overline{\Lambda}$ acts on Y_ϵ by biholomorphisms preserving t, with $\Psi_{\bar\lambda}(O_{\hat P}) = O_{\widehat{P+B_0\bar\lambda}}$ (as $\tau_{\bar\lambda}$ preserves each $T_{N'}$-orbit). Charts are covered by Lemma 4.4(ii).

Step 1 *(freeness).* Let $\bar\lambda \neq 0$. On $T_{N'} \cap Y_\epsilon$, $\Psi_{\bar\lambda}(x,t) = (x,t)$ forces $c_{\bar\lambda}(t) t^{B_0\bar\lambda} = (1,1)$ by (4.2); taking $\log|\cdot|$ and using $\log|c_{\bar\lambda}(t)| = \mathsf{R}(t)\bar\lambda$ gives $\log|t| \cdot B_0\bar\lambda + \mathsf{R}(t)\bar\lambda = 0$. But $|\log|t||\,|B_0\bar\lambda| = |\log|t||\,|\bar\lambda| > 2\|\mathsf{R}\|_\infty|\bar\lambda| \geq |\mathsf{R}(t)\bar\lambda|$ by (4.1) and the choice of ϵ, a contradiction. On $t^{-1}(0) \cap Y_\epsilon$, $\Psi_{\bar\lambda}$ maps $O_{\hat P}$ to $O_{\widehat{P+B_0\bar\lambda}}$ and $P + B_0\bar\lambda \neq P$ since P is bounded and $B_0\bar\lambda \neq 0$.

Step 2 *(which charts a translate can meet).* Put $\eta_0 := \log 2/|\log\epsilon|$ and, for $0 < |t| < \epsilon$,

$$B_t := B_0 + \frac{\mathsf{R}(t)}{\log|t|}, \qquad \|B_t - B_0\| \leq \frac{\|\mathsf{R}\|_\infty}{|\log\epsilon|} \leq \tfrac{1}{2}, \tag{4.4}$$

so $|B_t\bar\lambda| \geq \frac{1}{2}|\bar\lambda|$, $\|B_t\| \leq \frac{3}{2}$, and B_t is invertible. For triangles σ, σ' the set $L(\sigma,\sigma') := \{\bar\lambda : \Psi_{\bar\lambda}\mathcal{U}_\sigma(2,\epsilon) \cap \mathcal{U}_{\sigma'}(2,\epsilon) \neq \emptyset\}$ is finite. Indeed $T_{N'} \cap Y_\epsilon$ is dense in Y_ϵ (in a chart $Y_\epsilon \cap \mathcal{U}_\sigma = \{|z_0z_1z_2| < \epsilon\}$, with torus $\{z_0z_1z_2 \neq 0\}$) and $\Psi_{\bar\lambda}$-invariant, so the open set $\Psi_{\bar\lambda}\mathcal{U}_\sigma(2,\epsilon) \cap \mathcal{U}_{\sigma'}(2,\epsilon)$, if nonempty, contains $q = \Psi_{\bar\lambda}(p)$ with p a torus point of $\mathcal{U}_\sigma(2,\epsilon)$. With $t = t(p) = t(q)$, (4.2) gives

$$y(q) = y(p) + B_0\bar\lambda + \frac{\mathsf{R}(t)\bar\lambda}{\log|t|} = y(p) + B_t\bar\lambda, \tag{4.5}$$

while Lemma 4.4(i) gives $y(p) \in \sigma^{(\eta_0)}$, $y(q) \in \sigma'^{(\eta_0)}$. Hence

$$\tfrac{1}{2}|\bar\lambda| \leq |B_t\bar\lambda| = |y(q) - y(p)| \leq d(\sigma,\sigma') := \sup\{|a-b| : a \in \sigma'^{(\eta_0)}, b \in \sigma^{(\eta_0)}\} < \infty.$$

Step 3 *((a) and (b)).* Let $K \subset Y_\epsilon$ be compact; by Lemma 4.4(ii), $K \subset \bigcup_{\sigma\in\mathcal{F}} \mathcal{U}_\sigma(2,\epsilon)$ with $\mathcal{F}$ finite, and $\Psi_{\bar\lambda}K \cap K \neq \emptyset$ implies $\bar\lambda \in L(\sigma,\sigma')$ for some $(\sigma,\sigma') \in \mathcal{F}\times\mathcal{F}$: this is (a).

We deduce (b) from (a) and freeness directly; Y_ϵ is locally compact Hausdorff, and pr is open because $\mathrm{pr}^{-1}(\mathrm{pr}(V)) = \bigcup_{\bar\lambda} \Psi_{\bar\lambda}V$ is open for V open. *Covering:* let $p \in Y_\epsilon$ and let K be a compact neighbourhood of p; by (a) the set $F := \{\bar\lambda \neq 0 : \Psi_{\bar\lambda}K \cap K \neq \emptyset\}$ is finite, and $\Psi_{\bar\lambda}p \neq p$ for $\bar\lambda \in F$ by Step 1, so by Hausdorffness we may choose, for each $\bar\lambda \in F$, disjoint open sets $A_{\bar\lambda} \ni p$ and $A'_{\bar\lambda} \ni \Psi_{\bar\lambda}p$. Then $V := \operatorname{int} K \cap \bigcap_{\bar\lambda\in F}\big(A_{\bar\lambda} \cap \Psi_{\bar\lambda}^{-1}A'_{\bar\lambda}\big)$ is an open neighbourhood of p with $\Psi_{\bar\lambda}V \cap V = \emptyset$ for all $\bar\lambda \neq 0$ (for $\bar\lambda \in F$ by construction, for $\bar\lambda \notin F$ because $V \subset K$). Hence $\mathrm{pr}|_V$ is an open continuous injection, i.e. a homeomorphism onto the open set $\mathrm{pr}(V)$, whose preimage is the disjoint union $\bigsqcup_{\bar\lambda} \Psi_{\bar\lambda} V$: pr is a covering. *Hausdorff quotient:* let $p, q \in Y_\epsilon$ lie in distinct orbits and let K_p, K_q be compact neighbourhoods; by (a) applied to $K_p \cup K_q$ the set $F' := \{\bar\lambda : \Psi_{\bar\lambda} K_p \cap K_q \neq \emptyset\}$ is finite, and $\Psi_{\bar\lambda} p \neq q$ for every $\bar\lambda$, so as above we find open $V_p \subset \operatorname{int} K_p$, $V_q \subset \operatorname{int} K_q$ containing p, q with $\Psi_{\bar\lambda} V_p \cap V_q = \emptyset$ for all $\bar\lambda \in F'$, hence for all $\bar\lambda$; then $\mathrm{pr}(V_p), \mathrm{pr}(V_q)$ are disjoint open neighbourhoods of $\mathrm{pr}(p), \mathrm{pr}(q)$.

Since the deck transformations are biholomorphic, N_0 is a complex manifold with pr holomorphic; it is second countable (image of a second countable space under an open map) and connected because Y_ϵ is ($\mathcal{U}_\sigma \cap Y_\epsilon = \{|z_0 z_1 z_2| < \epsilon\}$ is star-shaped about the origin and meets the connected $T_{N'} \cap Y_\epsilon$). Finally t is invariant, so descends to f_0.

Step 4 (properness; (c)). Fix $0 < \epsilon_1 < \epsilon$, let $\mathcal{F}_0 := \{\sigma : \sigma \cap \bar B(0, \sqrt{2}) \neq \emptyset\}$ (finite) and $K_0 := \bigcup_{\sigma \in \mathcal{F}_0} (\bar{\mathcal{U}}_\sigma(1) \cap \{|t| \le \epsilon_1\})$, a compact subset of Y_ϵ. We claim $\{|t| \le \epsilon_1\} \subset \overline{\Lambda} \cdot K_0$. For a torus point p with $0 < |t(p)| \le \epsilon_1$ put $u := B_{t(p)}^{-1} y(p)$ and choose $\bar\lambda \in \mathbb{Z}^2$ with $|u + \bar\lambda|_\infty \le \frac{1}{2}$ ($\bar\lambda$ ranging over the lattice $\overline{\Lambda} \cong \mathbb{Z}^2$); by (4.5) and (4.4),

$$|y(\Psi_{\bar\lambda} p)| = |B_{t(p)}(u + \bar\lambda)| \le \|B_{t(p)}\| \cdot \tfrac{\sqrt{2}}{2} \le \tfrac{3}{2\sqrt{2}} < \sqrt{2},$$

so $\Psi_{\bar\lambda} p \in \bar{\mathcal{U}}_\sigma(1)$ for any $\sigma \ni y(\Psi_{\bar\lambda} p)$ by Lemma 4.4(i), and such $\sigma \in \mathcal{F}_0$. If $p \in O_{\hat P} \subset t^{-1}(0)$, pick a vertex v_0 of P and $\bar\lambda$ with $B_0 \bar\lambda = -v_0$ (possible, again, by the *equality* $B_0 \overline{\Lambda} = \mathbb{Z}^2$ of Lemma 4.3(i)); then $\Psi_{\bar\lambda} p$ lies in the orbit $O_{\widehat{P - v_0}}$, whose cell $P - v_0$ has 0 as a vertex, hence by Lemma 4.4(ii) it lies in some $\bar{\mathcal{U}}_\sigma(1)$ with $\sigma \supseteq P - v_0$, so that $\sigma \ni 0$, and every such σ meets $\bar B(0, \sqrt{2})$. Therefore $f_0^{-1}(\bar\Delta_{\epsilon_1}) = \mathrm{pr}(t^{-1}(\bar\Delta_{\epsilon_1})) = \mathrm{pr}(K_0)$ is compact; any compact $K \subset \Delta_\epsilon$ lies in some $\bar\Delta_{\epsilon_1}$ and $f_0^{-1}(K)$ is closed in the compact Hausdorff $\mathrm{pr}(K_0)$, so f_0 is proper. Surjectivity is clear, and submersivity over $\Delta_\epsilon \setminus \{0\}$ follows from $t^{-1}(\mathbb{C}^*) = T_{N'}$.

For $t_0 \neq 0$, $f_0^{-1}(t_0) = (\{t_0\} \times (\mathbb{C}^*)^2)/\overline{\Lambda}$ by Lemma 4.2(ii), the action being multiplication by $c_{\bar\lambda}(t_0) t_0^{B_0 \bar\lambda}$. Choosing s with $e^{2\pi i s} = t_0$, (P2) gives

$$c_{\bar\lambda}(t_0)\, t_0^{B_0 \bar\lambda} = \exp\big(2\pi i\, (s B_0 + C(t_0)) \bar\lambda\big) = \exp\big(2\pi i\, Z(s) \bar\lambda\big) \tag{4.6}$$

(the left side is single valued since $B_0 \bar\lambda \in \mathbb{Z}^2$), so the covering $\exp(2\pi i \cdot) : \mathbb{C}^2 \to (\mathbb{C}^*)^2$, with deck group $\mathbb{Z}^2 = \Pi \Lambda_{\mathrm{tor}}$, identifies $f_0^{-1}(t_0)$ with $\mathbb{C}^2/(\mathbb{Z}^2 \oplus Z(s)\mathbb{Z}^2) = \mathbb{C}^2/\Pi(z)\Lambda = \mathbb{C}^2/L(z)$. This is a compact torus: by (4.4), $\operatorname{Im} Z(s) = \operatorname{Im}(s) B_0 + \operatorname{Im} C(t_0) = -\frac{\log|t_0|}{2\pi} B_{t_0}$ is invertible, so $e_1, e_2, Z(s)\bar\gamma, Z(s)\bar u$ are $\mathbb{R}$-independent (consistently with the nondegeneracy $D < 0$ of Theorem 3.4).

Step 5 ((d), (e)). Since pr is a local biholomorphism carrying t to f_0 and $t^{-1}(0)$ onto W, it suffices to produce the asserted normal forms for t at a point $\tilde p \in t^{-1}(0) \cap Y_\epsilon$ lying over p, the number of local branches of W at p being the number of divisors E_v through $\tilde p$ (the covering is biholomorphic near $\tilde p$). Choose a triangle $\sigma = \{v_0, v_1, v_2\}$ with $\tilde p \in \mathcal{U}_\sigma$ (Lemma 4.4(ii)) and the coordinates (z_0, z_1, z_2) of Lemma 4.2(iii), so that $t = z_0 z_1 z_2$ and $E_{v_i} \cap \mathcal{U}_\sigma = \{z_i = 0\}$. Let $I := \{i : z_i(\tilde p) = 0\}$; since $t(\tilde p) = 0$, $I \neq \emptyset$, and $\tilde p$ lies on exactly the $|I|$ divisors E_{v_i}, $i \in I$ (by the orbit–cone correspondence, as recorded in the proof of Lemma 4.2(i), no E_v with v not a vertex of σ meets $\mathcal{U}_\sigma$). On the open neighbourhood $\{z_j \neq 0\ \forall j \notin I\}$ of $\tilde p$ the function $u := \prod_{j \notin I} z_j$ is a nowhere vanishing holomorphic unit, and replacing one of the coordinates z_i, $i \in I$, by $u\, z_i$ and translating the z_j, $j \notin I$, to be centred at $\tilde p$ yields holomorphic coordinates centred at $\tilde p$ in which $t = \prod_{i \in I} z_i$. Thus t, hence f_0, is z_0, $z_0 z_1$ or $z_0 z_1 z_2$ according as $|I| = 1, 2, 3$, i.e. according as p lies on one, two or three branches of W. In particular $f_0^*(0) = W$ with W reduced — as already follows from Lemma 4.2(ii), the rays of $\mathfrak{F}$ all having height 1 — and W is a normal crossings surface; by Proposition 4.6 its singular locus is $\mathsf{D} = \mathsf{D}_1 \cup \mathsf{D}_2 \cup \mathsf{D}_3$ (the case $|I| = 2$) and its triple points are exactly P, Q (the case $|I| = 3$). Connectedness of W is Proposition 4.6(i).

For (e): on $\mathcal{U}_\sigma$ the bases (m_0, m_1, m_2) and $((1,0,0), (0,1,0), (0,0,1))$ of M' differ by a matrix of determinant ± 1, so $\bigwedge_i dz_i/z_i = \pm \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dt}{t}$ and, using $t = z_0 z_1 z_2$,

$$\Omega = \pm\, t \cdot \frac{dz_0 \wedge dz_1 \wedge dz_2}{z_0 z_1 z_2} = \pm\, dz_0 \wedge dz_1 \wedge dz_2,$$

a nowhere vanishing holomorphic 3-form on Y. It is invariant: by (4.2), $\Psi_{\bar\lambda}^* \frac{dx_i}{x_i} = \frac{dx_i}{x_i} + \alpha_i\, dt$ and $\Psi_{\bar\lambda}^* dt = dt$, and the extra terms contain dt twice; by density and continuity $\Psi_{\bar\lambda}^* \Omega = \Omega$ on Y_ϵ, so Ω trivialises K_{N_0}. $\square$

4.6 The central fibre W

Proposition 4.6. Let $\nu := \mathrm{pr}|_{E_0} : E_0 \to W$ and $C_v := E_0 \cap E_v$ the six lines of the hexagon on $E_0 \cong \mathrm{dP}_6$.

(i) ν is surjective, finite, and injective on $E_0 \setminus \bigcup_v C_v$; hence W is irreducible and connected, and ν is its normalisation. (Here W is reduced by Lemma 4.2(ii): all rays of $\mathfrak{F}$ lie at height 1, so $t^{-1}(0)$ is reduced upstairs, and pr is a local biholomorphism.)

(ii) For each pair $\{v, -v\}$, $v \in \{e_1, e_2, e_1 - e_2\}$, ν identifies C_v with C_{-v} by $\Psi_{\bar\lambda}|_{C_v}$, $B_0\bar\lambda = -v$, and makes no other identification.

(iii) The double locus $\mathsf{D} \subset W$ consists of three smooth rational curves $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$; any two meet exactly in the two triple points P, Q of W, the images of the two $\overline{\Lambda}$-orbits of triangles of $\mathcal{T}_{A_2}$, through each of which all three pass. The six hexagon vertices of E_0 map three-to-one onto $\{P, Q\}$, alternately around the hexagon.

(iv) $e(W) = 2$.

Proof. (i),(ii) By Theorem 4.5, $W = t^{-1}(0)/\overline{\Lambda}$, with $\Psi_{\bar\lambda}(O_{\hat P}) = O_{\widehat{P+B_0\bar\lambda}}$ and $B_0\overline{\Lambda} = \mathbb{Z}^2$. Every cell has a vertex and the vertices form one $\mathbb{Z}^2$-orbit, so every $O_{\hat P} \subset t^{-1}(0)$ is equivalent to one with $0 \in P$, i.e. contained in $E_0 = \bigcup_{P \ni 0} O_{\hat P}$: ν is surjective, and its fibres have at most three points (three cells containing 0 lie in one $\mathbb{Z}^2$-orbit only if they are triangles), so ν is finite, E_0 being compact. Two points of E_0 are identified only if their cells $P, P' \ni 0$ satisfy $P' = P + w$, $w \in \mathbb{Z}^2 \setminus \{0\}$, which forces either $P = \{0, v\}$, $P' = \{0, -v\}$, $w = -v$, or P, P' triangles containing 0; in particular ν is injective on the open orbit. This is (ii) and the identification of the hexagon vertices: the two $\mathbb{Z}^2$-orbits of triangles are the "lower" ones $\{v, v + e_1, v + e_2\}$ and the "upper" ones $\{v + e_1, v + e_2, v + e_1 + e_2\}$, and the six triangles containing 0 alternate between them around the hexagon, by the cyclic list in the proof of Lemma 4.2(iv). Since E_0 is smooth, compact and irreducible with ν finite and generically injective, ν is the normalisation of W (W being reduced by Theorem 4.5(d), whose proof of reducedness does not use the present Proposition), which is therefore irreducible and connected.

(iii) Each $C_v \cong \mathbb{P}^1$ maps isomorphically onto its image (no $\Psi_{\bar\lambda}$, $\bar\lambda \neq 0$, preserves the edge $\{0, v\}$, and the two triangles on $\{0, v\}$ lie in different $\mathbb{Z}^2$-classes), and C_v, C_{-v} have the same image; this gives three smooth rational curves $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ (images of $C_{e_1}, C_{e_2}, C_{e_1-e_2}$) whose union is the locus where ν is not injective, i.e. D. Each C_v contains exactly two hexagon vertices, one of each class, so each D_i passes through P and Q; conversely two distinct D_i meet only where ν identifies points of two non-opposite lines of E_0, i.e. only at P, Q.

(iv) W is stratified by the images of the $T_{N'}$-orbits of $t^{-1}(0)$, one for each $\overline{\Lambda}$-orbit of cells: one stratum $\cong (\mathbb{C}^*)^2$, three $\cong \mathbb{C}^*$ (edge directions $e_1, e_2, e_1 - e_2$), and two points. By additivity of the topological Euler characteristic over finite stratifications by locally closed subspaces, $e(W) = 0 + 3 \cdot 0 + 2 = 2$; equivalently $e(W) = (6 - 6) + (3 \cdot 2 - 2 \cdot 2) = 2$ from $e(\mathrm{dP}_6) = 6$. In sum, W is a single copy of dP_6 with opposite sides of its hexagon identified, with double locus the union of the three rational curves $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$ through the two triple points P, Q. □

4.7 The tautological identification at p_0

Recall from §3 that $J|_{0<|t_c|<\epsilon} = (U_\epsilon \times \mathbb{C}^2)/G$, where U_ϵ is the cusp neighbourhood over $\{0 < |t_c| < \epsilon\}$ and G is generated by $\mathsf{t}_\lambda(z, \zeta) = (z, \zeta + \Pi(z)\lambda)$, $\lambda \in \Lambda$, and $\tilde g_0$. The cocycle relation $\Pi(gz) = R_g \Pi(z) M_g$ gives $R_g \Pi(z) = \Pi(gz) M_g^{-1}$, hence

$$\tilde g\, \mathsf{t}_\lambda = \mathsf{t}_{M_g^{-1}\lambda}\, \tilde g \qquad (\lambda \in \Lambda), \tag{4.7}$$

since $\tilde g(z, \zeta + \Pi(z)\lambda) = (gz, R_g\zeta + \Pi(gz) M_g^{-1}\lambda)$. As Λ_{tor} is $M_{g_0}^{\pm 1}$-invariant (it is the monodromy-invariant sublattice, and $(M_0 - I)\Lambda = \Lambda_{\mathrm{tor}}$), (4.7) shows that the subgroup $H := \langle \mathsf{t}_\lambda\ (\lambda \in \Lambda_{\mathrm{tor}}), \tilde g_0 \rangle$ is normal in G, with $G/H \cong \Lambda/\Lambda_{\mathrm{tor}} = \overline{\Lambda}$.

Proposition 4.7. Define $E_0^{\exp} : U_\epsilon \times \mathbb{C}^2 \to T_{N'}$ by $E_0^{\exp}(z,\zeta) := \big(e^{2\pi i \zeta_1}, e^{2\pi i \zeta_2}, e^{2\pi i s(z)}\big)$ (the map E_0 of page 1). Then:

(i) $E_0^{\exp}$ is a surjective local biholomorphism onto $Y_\epsilon \cap T_{N'}$ whose fibres are the H-orbits;

(ii) $E_0^{\exp} \circ \mathrm{t}_\lambda = \Psi_{\bar\lambda} \circ E_0^{\exp}$ for $\lambda \in \Lambda$, $\bar\lambda = \lambda + \Lambda_{\mathrm{tor}}$;

(iii) consequently $E_0^{\exp}$ induces a biholomorphism

$$G_0 :\ J|_{0<|t_c|<\epsilon} \ \xrightarrow{\ \sim\ }\ (Y_\epsilon \cap T_{N'})/\overline{\Lambda} = N_0 \setminus W$$

over $\{0 < |t_c| < \epsilon\}$, carrying the marking $H_1(\text{fibre}) = \Lambda$ of §3 to the tautological one: $\Lambda_{\mathrm{tor}} = H_1((\mathbb{C}^*)^2)$ and $\Lambda/\Lambda_{\mathrm{tor}} = \overline{\Lambda} =$ the deck group of $f_0^{-1}(t_0)$;

(iv) G_0 carries the 0-section of J to the section $x = (1,1)$, which extends to a holomorphic section ς of f_0 over Δ_ϵ meeting W in a point of $W \setminus \mathrm{D}$. Moreover ς is injective and $\varsigma(\Delta_\epsilon)$ is a closed complex submanifold of N_0, biholomorphic to Δ_ϵ via f_0.

Proof. (i) By (P1), s maps U_ϵ biholomorphically onto $\{\operatorname{Im} s > \frac{1}{2\pi}\log(1/\epsilon)\}$, which $e^{2\pi i s}$ maps onto $\{0 < |t_c| < \epsilon\}$ with $s(z) - s(z') \in \mathbb{Z}$ the only coincidences; and $T_{N'} \cap Y_\epsilon = \{p \in T_{N'} : 0 < |t(p)| < \epsilon\}$ by Lemma 4.2(ii). So $E_0^{\exp}$ is a surjective local biholomorphism, and $E_0^{\exp}(z,\zeta) = E_0^{\exp}(z',\zeta')$ precisely when $s(z) - s(z') = n \in \mathbb{Z}$ and $\zeta - \zeta' \in \mathbb{Z}^2$. By (P1) $z' = g_0^n z$, by (P3) $\tilde g_0^n(z,\zeta) = (g_0^n z, \zeta)$, and $\mathbb{Z}^2 = \Pi(z)\Lambda_{\mathrm{tor}}$ since $\Pi\hat w = e_1$, $\Pi\hat\delta = e_2$; so the fibres are the H-orbits.

(ii) Write $\lambda = a\hat\gamma + b\hat u + (\text{element of } \Lambda_{\mathrm{tor}})$, $\bar\lambda = (a,b)^t$, so $\Pi(z)\lambda \equiv Z(z)\bar\lambda \pmod{\mathbb{Z}^2}$. With $t_c = e^{2\pi i s(z)}$, (4.6) gives

$$\exp\big(2\pi i(\zeta + \Pi(z)\lambda)\big) = c_{\bar\lambda}(t_c)\, t_c^{B_0\bar\lambda} \cdot \exp(2\pi i \zeta),$$

which by (4.2) is the first two coordinates of $\Psi_{\bar\lambda}(E_0^{\exp}(z,\zeta))$; the last is unchanged.

(iii) By (i),(ii), $E_0^{\exp}$ descends to a biholomorphism $(U_\epsilon \times \mathbb{C}^2)/H \to Y_\epsilon \cap T_{N'}$ intertwining the residual $\overline{\Lambda}$-actions, hence to G_0; both maps commute with the projections to the punctured disc, and the statement on markings is the content of (i),(ii).

(iv) $E_0^{\exp}(z,0) = (1,1,e^{2\pi i s(z)})$. For $\sigma = \{0, e_1, e_2\}$ a direct computation gives $m_0 = (-1,-1,1)$, $m_1 = (1,0,0)$, $m_2 = (0,1,0)$, i.e. $(z_0,z_1,z_2) = (t/(x_1x_2), x_1, x_2)$; so $\{(1,1)\} \times \Delta_\epsilon = \{(t,1,1) : |t| < \epsilon\}$ is a closed holomorphic section of t on Y_ϵ meeting $t^{-1}(0)$ at $(0,1,1)$ in the open orbit $O_{\hat 0} \subset E_0$. Two of its points in one fibre of f_0 would differ by some $\Psi_{\bar\lambda}$, hence lie over the same value of t because $t \circ \Psi_{\bar\lambda} = t$, and therefore coincide, the section meeting each fibre of t in exactly one point; so it descends to a section ς with $\varsigma(0) \in \nu(O_{\hat 0}) = W \setminus \mathrm{D}$ by Proposition 4.6. Being a section of f_0, ς is injective with $f_0 \circ \varsigma = \mathrm{id}$, so it is a biholomorphism of Δ_ϵ onto its image; the image is closed in N_0 because ς is proper over Δ_ϵ (f_0 being proper, $\varsigma(\bar\Delta_{\epsilon_1})$ is compact for every $\epsilon_1 < \epsilon$). $\square$

Corollary 4.8. $\pi_1(Y_\epsilon) = 1$; hence pr is the universal covering and $\pi_1(N_0) \cong \overline{\Lambda} \cong \mathbb{Z}^2$. Moreover:

(i) for $0 < |t_0| < \epsilon$ the map $\pi_1(f_0^{-1}(t_0)) \to \pi_1(N_0)$ is the projection $\Lambda \to \overline{\Lambda}$ (in the marking of Proposition 4.7); in particular the vanishing cycles Λ_{tor} die in $\pi_1(N_0)$;

(ii) the loop $\varsigma(\{|t_c| = \rho\})$, $0 < \rho < \epsilon$, is null-homotopic in N_0.

Proof. First, $\pi_1(Y) = 1$. As the statement of [Ful93, §3.2] is made for finite fans, we reduce to that case: for $n \in \mathbb{N}$ let $\mathfrak{F}_n \subset \mathfrak{F}$ be the (finite) subfan of the cones $\hat P$ over the cells P of $\mathcal{T}_{A_2}$ contained in the disc $\bar B(0,n)$, together with $\{0\}$. Then $Y_{\mathfrak{F}_n}$ is an open subset of Y, $Y = \bigcup_{n\ge 1} Y_{\mathfrak{F}_n}$ is an increasing union of open sets, and each $Y_{\mathfrak{F}_n}$ is a smooth toric variety of finite type whose rays $\mathbb{R}_{\ge 0}(v,1)$, $|v| \le n$, span N' over $\mathbb{Z}$ (already $v = 0, e_1, e_2$ suffice, for $n \ge 1$); hence $\pi_1(Y_{\mathfrak{F}_n}) = N'/\langle (v,1)\rangle = 1$ by [Ful93, §3.2]. Since π_1 commutes with increasing unions of open sets (every loop and every null-homotopy has compact image, hence lies in some $Y_{\mathfrak{F}_n}$), $\pi_1(Y) = \varinjlim_n \pi_1(Y_{\mathfrak{F}_n}) = 1$.

For $0 < u \le 1$ let $\vartheta_u = (1,1,u) \in T_{N'}$; then $t(\vartheta_u \cdot p) = u\, t(p)$, so $\vartheta_u(Y_\epsilon) \subset Y_\epsilon$. A loop γ in Y_ϵ bounds a disc $h : D^2 \to Y$; for $u < \epsilon / \sup_{D^2} |t \circ h|$ the disc $\vartheta_u \circ h$ lies in Y_ϵ and bounds $\vartheta_u \circ \gamma$, which is homotopic to

γ in Y_ϵ through $\vartheta_{u'} \circ \gamma$, $u' \in [u, 1]$. Hence $\pi_1(Y_\epsilon) = 1$ and $\pi_1(N_0)$ is the deck group $\overline{\Lambda}$. (i) Restrict pr over t_0: it is the covering $(\mathbb{C}^*)^2 \to f_0^{-1}(t_0)$ with deck group $\overline{\Lambda}$, and $\pi_1((\mathbb{C}^*)^2) = \Lambda_{\mathrm{tor}}$ maps to 1 in $\pi_1(Y_\epsilon) = 1$. (ii) The loop lifts to $\{(1,1)\} \times \{|t| = \rho\} \subset Y_\epsilon$. $\square$

Remark 4.9. The only properties of the data entering §§4.2–4.7 are: (a) $B_0 \in M_2(\mathbb{Z})$ with $\det B_0 \neq 0$, used through $B_0\overline{\Lambda} \subset \mathbb{Z}^2 = \mathrm{Vert}(\mathcal{T}_{A_2})$ and through $|B_0\bar{\lambda}| \geq c_{B_0}|\bar{\lambda}|$, $c_{B_0} > 0$ (Steps 1,2); $|\det B_0| = 1$, i.e. the equality $B_0\overline{\Lambda} = \mathbb{Z}^2$, is used in addition in Lemma 4.3(i), in the boundary case of Step 4 of Theorem 4.5 (solving $B_0\bar{\lambda} = -v_0$ for an arbitrary vertex v_0), and in Proposition 4.6; the torus case of Step 4 uses only $\|B_t - B_0\| \leq \frac{1}{2}$ and the invertibility of B_t (if one only assumes $B_0\overline{\Lambda} \subset \mathbb{Z}^2$ of finite index, the boundary case of Step 4 still goes through with $\frac{1}{2}$ replaced by a constant depending on B_0, namely the covering radius of $B_0\overline{\Lambda}$ in the sup-norm, and with $\bar{B}(0, \sqrt{2})$ enlarged accordingly); in Proposition 4.6 the action on vertices must be simply transitive, so that W is irreducible with normalisation a single dP_6 (for $|\det B_0| = d$ one would get d components and $e(W) = 2d$); (b) holomorphy of C on the full disc $\{|t_c| < r\}$, used twice: through the bound $\|\mathsf{R}\|_\infty < \infty$ (Steps 1,2,4), and — through holomorphy of C at $t_c = 0$ — to make $\tau_{\bar{\lambda}}$, hence $\Psi_{\bar{\lambda}}$, holomorphic across $t = 0$ in Lemma 4.3(iii). No symmetry or positivity of B_0 is used or available (B_0 maps $\overline{\Lambda}$ to the different lattice Λ_{tor}, and the family of §3 carries no polarisation). Re-indexing $\bar{\lambda} \mapsto -\bar{\lambda}$ shows that $\{\Psi_{\bar{\lambda}}^{(B_0,C)}\}$ and $\{\Psi_{\bar{\lambda}}^{(-B_0,-C)}\}$ are the same subgroup of $\mathrm{Aut}(Y_\epsilon)$; substituting $\mu := B_0\bar{\lambda}$ rewrites (4.2) as $x \mapsto c'_\mu(t)t^\mu x$ with $c'_\mu := c_{B_0^{-1}\mu}$, so N_0 is a two-dimensional Tate-type degeneration twisted by the holomorphic matrix CB_0^{-1}. Here $\mathcal{T}_{A_2}$ is prescribed, $\mathbb{Z}^2$-periodic, locally finite with bounded unimodular cells, all of whose rays $\hat{v}$ lie at height 1 (whence the reducedness of $t^{-1}(0)$, Lemma 4.2(ii)), and these are the only features of it that the above arguments require; correspondingly f_0 is not asserted to be projective and the fibres $f_0^{-1}(t_0)$ need not be algebraic — as they must not be, for the global manifold X of §6.

5 The fillings at p_1 and p_2: logarithmic transforms.

Conventions for this section

Throughout, $j \in \{1,2\}$ is fixed, $m_1 = 3$, $m_2 = 4$, and all notation is that of §2 and §3; for $S \in GL(V)$ we write $V^S := \ker(S - I)$, and $\Lambda^{A_j} = \ker(A_j - I)$. The subscripted $\zeta_j = e^{-2\pi i/m_j}$ is a root of unity, the unsubscripted ζ the fibre coordinate on $\mathbb{C}^2$. The construction modifies the family J only over a disc $D_j \ni p_j$, leaving $J|_{B^\circ}$ untouched.

5.1 The local family and its tautological deck transformation

Since Δ acts properly discontinuously on $\mathfrak{h}_z$ with stabiliser $\langle g_j \rangle$ at z_j, we may choose the disc $\Delta_j \ni z_j$ of §3 so small that $g_j\Delta_j = \Delta_j$, that the linearising coordinate s_j maps Δ_j biholomorphically onto a *round* disc $\{|s| < r_j\}$ with $s_j(z_j) = 0$ and $s_j \circ g_j = \zeta_j s_j$, and that $g\Delta_j \cap \Delta_j = \emptyset$ for $g \in \Delta \smallsetminus \langle g_j \rangle$. We fix once and for all the following terminology, used here and in §6–§7.

Definition 5.1. A *linearising pair* (Δ_j, s_j) at z_j consists of a g_j-invariant simply connected open neighbourhood $\Delta_j \ni z_j$ in $\mathfrak{h}_z$ with $g\Delta_j \cap \Delta_j = \emptyset$ for $g \in \Delta \smallsetminus \langle g_j \rangle$, together with a coordinate s_j on Δ_j (a holomorphic injection) such that

$$s_j(z_j) = 0, \qquad s_j \circ g_j = \zeta_j\, s_j, \qquad \zeta_j = e^{-2\pi i/m_j}.$$

The pair is *round* if moreover $s_j(\Delta_j) = \{|s| < r_j\}$ for some $r_j > 0$.

By the previous paragraph we may and do choose (Δ_j, s_j) to be a round linearising pair; in particular Δ_j is *simply connected*. Simple connectedness is used in §6 (Proposition 6.3(c),(d)), where branches of log of nowhere-vanishing functions on Δ_j are taken, and it is retained as part of the convention. Beyond simple connectedness, nothing below uses the shape of Δ_j: every statement of this section and of §6 holds verbatim for an arbitrary linearising pair (Δ_j, s_j) in the sense of Definition 5.1, round or not. Then $\pi|_{\Delta_j}$

induces $\Delta_j/\langle g_j\rangle \xrightarrow{\sim} D_j := \pi(\Delta_j)$, the function t_j with $t_j \circ \pi = s_j^{m_j}$ is a coordinate on D_j centred at p_j, and $\pi^{-1}(D_j) = \bigsqcup g\Delta_j$ over $g\langle g_j\rangle \in \Delta/\langle g_j\rangle$.

Put $J_j' := \mathcal{T}|_{\Delta_j} = (\Delta_j \times \mathbb{C}^2)/\Lambda$, where $\lambda \in \Lambda$ acts by $t_\lambda(z,\zeta) = (z, \zeta + \Pi(z)\lambda)$. By Theorem 3.4, $L(z) = \Pi(z)\Lambda$ is a lattice in $\mathbb{C}^2$ for *every* $z \in \mathfrak{h}_z$, in particular at $z = z_j$; hence $J_j' \to \Delta_j$ is a proper holomorphic submersion with fibres complex 2-tori and with the zero section, and we write $A_0^{(j)} := \mathbb{C}^2/L(z_j)$ for its central fibre. Moreover, $\Delta_j \cap \Delta \cdot z_{j'} = \emptyset$ for $j' \neq j$: if $gz_{j'} \in \Delta_j$ for some $g \in \Delta$, then $gg_{j'}g^{-1}$ fixes the point $gz_{j'} \in \Delta_j$, while $gg_{j'}g^{-1} \notin \langle g_j\rangle$ (it has order $m_{j'} \neq m_j$), contradicting the convention that the only elements of Δ with fixed points in Δ_j are those of $\langle g_j\rangle$ (if $h \in \Delta \setminus \langle g_j\rangle$ fixed $x \in \Delta_j$ then $h\Delta_j \cap \Delta_j \ni x$). Hence $\Delta_j^* = \Delta_j \setminus \{z_j\} \subset \mathfrak{h}_z^\circ$. Since z_j is the only point of Δ_j over p_j, the definition $J = (\mathcal{T}|_{\mathfrak{h}_z^\circ})/\Delta$ gives

$$J|_{D_j^*} = \big(J_j'|_{\Delta_j^*}\big)/\langle \tilde{g}_j^0\rangle, \qquad \tilde{g}_j^0 := \widetilde{g}_j\big|_{\Delta_j \times \mathbb{C}^2}, \tag{5.1}$$

the quotient being by a free $\mathbb{Z}/m_j$-action (Δ acts freely on $\mathfrak{h}_z^\circ$).

Lemma 5.2. For $b \in \Lambda \otimes \mathbb{R}$ let $\tilde{g}_j^b(z,\zeta) := \big(g_jz,\ R_{g_j}(z)\zeta + \Pi(g_jz)b\big)$ on $\Delta_j \times \mathbb{C}^2$. Then:

(i) $\tilde{g}_j^b$ is a biholomorphism covering g_j with $\tilde{g}_j^b \circ t_\lambda = t_{A_j\lambda} \circ \tilde{g}_j^b$; hence it descends to a biholomorphism of J_j' over g_j, again written $\tilde{g}_j^b$.

(ii) $\Xi : \Delta_j \times (\Lambda \otimes \mathbb{R})/\Lambda \to J_j'$, $\Xi(z, x + \Lambda) = [(z, \Pi(z)x)]$, is a fibre-preserving real-analytic diffeomorphism, and with $\Sigma_k := I + A_j + \cdots + A_j^{k-1}$,

$$\Xi^{-1}(\tilde{g}_j^b)^k\, \Xi\,(z,x) = (g_j^k z,\ A_j^k x + \Sigma_k b).$$

(iii) $\tilde{g}_j^0$ is the deck transformation of (5.1), and $\tilde{g}_j^{b+\lambda} = \tilde{g}_j^b$ on J_j' for $\lambda \in \Lambda$.

Proof. By Theorem 3.4, $R_{g_j}(z)$ is holomorphic and invertible on Δ_j and $\Pi(g_jz) = R_{g_j}(z)\Pi(z)M_{g_j}$, i.e.

$$R_{g_j}(z)\Pi(z) = \Pi(g_jz)M_{g_j}^{-1} = \Pi(g_jz)A_j. \tag{5.2}$$

Hence $\tilde{g}_j^b(z, \zeta + \Pi(z)\lambda) = t_{A_j\lambda}\tilde{g}_j^b(z,\zeta)$, which is (i); the same computation with λ real gives the case $k = 1$ of (ii), and the general case follows by induction. Ξ is a diffeomorphism because $\Pi(z) : \Lambda \otimes \mathbb{R} \to \mathbb{C}^2$ is an $\mathbb{R}$-isomorphism for each z (Theorem 3.4). Part (iii) is the definition of $\widetilde{g}_j$ together with $\Pi(g_jz)\lambda \in L(g_jz)$. $\square$

Definition 5.3. For $v \in \Lambda^{A_j}$ put $\tilde{g}_j^{\log} := \tilde{g}_j^{v/m_j}$, i.e.

$$\tilde{g}_j^{\log}(z,\zeta) = \big(g_jz,\ R_{g_j}(z)\zeta + \Pi(g_jz)v/m_j\big),$$

which in the flat coordinates of Lemma 5.2(ii) is $(z,x) \mapsto (g_jz,\ A_jx + v/m_j)$. For the threefold X we take $v = v_j$, i.e. $v_1 = \varepsilon$, $v_2 = -\varepsilon'$.

Theorem 5.4. Let $v \in \Lambda^{A_j}$ satisfy $3 \nmid \gamma(v)$ if $j = 1$, resp. $\gamma(v)$ odd if $j = 2$ (e.g. $v = v_j$, with $\gamma(v_1) = 1$, $\gamma(v_2) = -1$). Then:

(i) $\tilde{g}_j^{\log}$ generates a group $\cong \mathbb{Z}/m_j$ acting freely and holomorphically on J_j'.

(ii) $N_j := J_j'/\langle \tilde{g}_j^{\log}\rangle$ is a complex threefold, $q_j : J_j' \to N_j$ is an unramified m_j-sheeted covering, and $s_j^{m_j}$ descends to a proper surjective holomorphic $f_j : N_j \to D_j$ with $f_j^*(p_j) = m_jS_j$, where

$$S_j := (f_j^{-1}(p_j))_{\mathrm{red}} = A_0^{(j)}/\langle x \mapsto A_jx + v/m_j\rangle$$

is a smooth compact complex surface; moreover

$$N_j \setminus S_j = \big(J_j'|_{\Delta_j^*}\big)/\langle \tilde{g}_j^{\log}\rangle.$$

(iii) S_j is a bielliptic surface with $H_1(S_j;\mathbb{Z}) \cong \mathbb{Z}^2$ torsion free, of Bagnera–de Franchis type $\mathbb{Z}/3 \times \mathbb{Z}/3$ for $j = 1$ and $\mathbb{Z}/4 \times \mathbb{Z}/2$ for $j = 2$; both K_{S_j} and $\mathcal{O}_{N_j}(S_j)|_{S_j}$ are torsion in $\mathrm{Pic}(S_j)$ of order exactly m_j.

(iv) $\sigma_j(z) = \frac{\log s_j(z)}{2\pi i}\Pi(z)v$ is a well-defined holomorphic section of J'_j over Δ_j^*, the fibrewise translation h_j by σ_j is a biholomorphism of $J'_j|_{\Delta_j^*}$ over Δ_j^*, and $h_j\,\tilde{g}_j^{\log}\,h_j^{-1} = \tilde{g}_j^0$. Consequently h_j descends to a biholomorphism over D_j^*,

$$G_j :\ N_j \smallsetminus S_j \ \xrightarrow{\ \sim\ }\ \big(J'_j|_{\Delta_j^*}\big)/\langle \tilde{g}_j^0\rangle = J|_{D_j^*}.$$

5.2 Order, fixed points, freeness

Lemma 5.5. (i) Let $A \in GL(\Lambda)$ have finite order, $b \in \Lambda \otimes \mathbb{Q}$, and let $\alpha(x) = Ax + b$ be the induced self-map of $(\Lambda \otimes \mathbb{R})/\Lambda$. Then α has a fixed point if and only if $\varphi(b) \in \mathbb{Z}$ for every $\varphi \in V = \mathrm{Hom}(\Lambda, \mathbb{Z})$ with $\varphi \circ A = \varphi$. (ii) $V^{T_1} = V^{T_1^2} = \langle \gamma, \psi_1\rangle$ with $\psi_1 := 2u + w + 3\delta$, and $V^{T_2} = V^{T_2^2} = V^{T_2^3} = \langle \gamma, \psi_2\rangle$ with $\psi_2 := u + w + 2\delta$.

Proof. (i) Put $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$, $W := \{\varphi \in V : \varphi \circ A = \varphi\}$. A fixed point exists iff $b \in (A - I)\Lambda_{\mathbb{R}} + \Lambda$, which forces $\varphi(b) \in \mathbb{Z}$ for $\varphi \in W$. Conversely let N be the order of A and $p := \frac{1}{N}\sum_{i<N} A^i$, a rational projector with image $U = \ker(A - I)_{\mathbb{R}}$ and kernel $U' = \mathrm{im}(A - I)_{\mathbb{R}}$, so $\Lambda_{\mathbb{R}} = U \oplus U'$. Every $\varphi \in W_{\mathbb{R}} := W \otimes \mathbb{R}$ kills U' and $\dim W_{\mathbb{R}} = \dim U$, so the pairing $U \times W_{\mathbb{R}} \to \mathbb{R}$ is perfect. The group $p(\Lambda) \subset U$ is a lattice (it spans U and lies in $\frac{1}{N}\Lambda \cap U$) and $\varphi(p\lambda) = \varphi(\lambda)$ for $\varphi \in W_{\mathbb{R}}$; hence the dual lattice of $p(\Lambda)$ is W and, by biduality, $p(\Lambda) = \{u \in U : \varphi(u) \in \mathbb{Z}\ \forall \varphi \in W\}$. If $\varphi(b) \in \mathbb{Z}$ for all $\varphi \in W$, then $\varphi(pb) = \varphi(b) \in \mathbb{Z}$, so $pb = p\lambda$ with $\lambda \in \Lambda$ and $b - \lambda \in U'$.

(ii) A direct computation with the matrices of §2 gives, for $x = a\gamma + bu + cw + d\delta$,

$$T_1 x = (a - 6c + 2d)\gamma + (-b + c + d)u + (-b + d)w + d\delta,$$
$$T_2 x = (a + 6b - 3d)\gamma + (-c + d)u + bw + d\delta,$$

so that $T_1 x = x$ iff $b = 2c$ and $d = 3c$, while $T_2 x = x$ iff $c = b$ and $d = 2b$. Hence $V^{T_1} = \langle \gamma, \psi_1\rangle$ and $V^{T_2} = V^{T_2^3} = \langle \gamma, \psi_2\rangle$. Also $V^{T_1^2} = V^{T_1^{-1}} = V^{T_1}$, and T_2^2 acts by

$$\gamma \mapsto \gamma, \quad u \mapsto 6\gamma - u, \quad w \mapsto -6\gamma - w, \quad \delta \mapsto u + w + \delta,$$

with fixed space $\{b = c,\ d = 2b\} = \langle \gamma, \psi_2\rangle$. □

Proposition 5.6. Let $v \in \Lambda^{A_j}$. Then $(\tilde{g}_j^{\log})^{m_j} = \mathrm{id}$ on J'_j, the order of $\tilde{g}_j^{\log}$ is exactly m_j, and for $0 < k < m_j$ the fixed points of $(\tilde{g}_j^{\log})^k$ lie in the central fibre $A_0^{(j)}$. Moreover $\langle \tilde{g}_j^{\log}\rangle \cong \mathbb{Z}/m_j$ acts freely on J'_j if and only if $3 \nmid \gamma(v)$ for $j = 1$, resp. $\gamma(v)$ is odd for $j = 2$; in particular the action is free for $v_1 = \varepsilon$ and $v_2 = -\varepsilon'$.

Proof. In flat coordinates $(\tilde{g}_j^{\log})^k$ is $x \mapsto A_j^k x + \Sigma_k(v/m_j)$, and $A_j v = v$ gives $\Sigma_k(v/m_j) = (k/m_j)v$. Since $A_j^{m_j} = I$ (as $T_j^{m_j} = I$), the m_j-th power is the translation by $v \in \Lambda$, i.e. the identity of J'_j; its order is exactly m_j because it covers g_j. As $(\tilde{g}_j^{\log})^k$ covers g_j^k and $s_j(g_j^k z) = \zeta_j^k s_j(z) \neq s_j(z)$ for $0 < k < m_j$ unless $s_j(z) = 0$, all fixed points lie over z_j.

For $\varphi \in V$ one has $\varphi \circ A_j^k = \varphi$ iff $\varphi \in V^{T_j^k}$ (because $A_j = (T_j^{-1})^t$), so by Lemma 5.5 the map $(\tilde{g}_j^{\log})^k$ has a fixed point iff $\varphi(\frac{k}{m_j}v) \in \mathbb{Z}$ for $\varphi \in \{\gamma, \psi_j\}$. Writing $v = a\varepsilon + c\hat{\delta} = a\hat{\gamma} + 2a\hat{u} - 4a\hat{w} + c\hat{\delta}$ for $j = 1$, resp. $v = a\varepsilon' + c\hat{\delta} = a\hat{\gamma} + 3a\hat{u} - 3a\hat{w} + c\hat{\delta}$ for $j = 2$, one gets

$$\gamma(v) = a, \qquad \psi_1(v) = 3c, \qquad \psi_2(v) = 2c.$$

So for $j = 1$ a fixed point exists iff $ka/3 \in \mathbb{Z}$ and $kc \in \mathbb{Z}$, i.e. iff $3 \mid ka$, which fails for $k = 1, 2$ precisely when $3 \nmid a$; for $j = 2$ it exists iff $ka/4 \in \mathbb{Z}$ and $kc/2 \in \mathbb{Z}$, which for $k = 2$ says a even and for $k = 1, 3$ requires $4 \mid a$, so the action is free iff $a = \gamma(v)$ is odd. Finally, for the choices made for X we have $\gamma(v_1) = \gamma(\varepsilon) = 1$, which is not divisible by 3, and $\gamma(v_2) = \gamma(-\varepsilon') = -1$, which is odd. □

This proves Theorem 5.4(i).

5.3 The threefold N_j, the multiple fibre and the surface S_j

Proof of Theorem 5.4(ii). A finite group acting freely and holomorphically on a complex manifold has a complex-manifold quotient with unramified projection [GrRe, Ch. V]; this gives N_j and q_j. The $\tilde{g}_j^{\log}$-invariant function $s_j^{m_j}$ descends to N_j and, read in the coordinate t_j, is f_j; it is surjective because s_j is. The projection $p_j : J_j' \to \Delta_j$ is proper (for $K \subset \Delta_j$ compact, the compact set $\{(z, \Pi(z)x) : z \in K,\ x \in [0,1]^4\}$ surjects onto $p_j^{-1}(K)$) and $\pi|_{\Delta_j}$ is finite, so $f_j^{-1}(K') = q_j\big(p_j^{-1}((\pi|_{\Delta_j})^{-1}(K'))\big)$ is compact for compact $K' \subset D_j$. The divisor of $f_j^* t_j$ pulls back under the étale q_j to that of $s_j^{m_j}$, which is m_j times the reduced smooth central fibre; hence $f_j^*(p_j) = m_j S_j$ with $S_j = q_j(A_0^{(j)})$, which by Lemma 5.2(ii) is the stated free quotient, a smooth compact surface. The description of $N_j \setminus S_j$ is immediate. $\square$

We write $\sigma : x \mapsto A_j x + v/m_j$ for the generating automorphism of $A_0^{(j)}$ and $G := \langle \sigma \rangle \cong \mathbb{Z}/m_j$.

Lemma 5.7. (i) $A_0^{(j)}$ is isogenous to a product of two elliptic curves; in particular it is abelian and S_j is projective. (ii) Let A be a compact complex torus, $G \subset \mathrm{Aut}(A)$ finite acting freely, $S = A/G$, and $\chi : G \to \mathbb{C}^*$ a character. Then $L_\chi := (A \times \mathbb{C})/G$, $g \cdot (x, \xi) = (gx, \chi(g)\xi)$, is trivial iff $\chi \equiv 1$; hence the order of L_χ in $\mathrm{Pic}(S)$ is the order of χ.

Proof. (i) Let $R_j := R_{g_j}(z_j)$; by (5.2) at $z = z_j$ it is the map induced by $A_j \otimes \mathbb{R}$ under the $\mathbb{R}$-isomorphism $\Pi(z_j)$. Hence $U := \ker(A_j - I)_{\mathbb{R}} = \ker(R_j - 1)$ and $U' := \mathrm{im}(A_j - I)_{\mathbb{R}} = \mathrm{im}(R_j - 1)$ are *complex* lines in $\mathbb{C}^2$ with $\mathbb{C}^2 = U \oplus U'$ (each of complex dimension 1, since $\mathrm{rk}\,\Lambda^{A_j} = 2$). Both are defined over $\mathbb{Q}$, so $\Lambda \cap U = \Lambda^{A_j}$ and $\Lambda \cap U' \supset (A_j - I)\Lambda$ are rank-2 lattices, and with $E := U/(\Lambda \cap U)$, $E' := U'/(\Lambda \cap U')$ the map $E \times E' \to A_0^{(j)}$ is a surjective homomorphism with finite kernel.

(ii) Sections of L_χ correspond to holomorphic $u : A \to \mathbb{C}$ with $u(gx) = \chi(g)u(x)$; such a u is constant, and a nowhere-vanishing one forces $\chi \equiv 1$. Apply this to $L_\chi^{\otimes k} = L_{\chi^k}$. $\square$

Proof of Theorem 5.4(iii). First homology. The universal cover of S_j is $\Lambda \otimes \mathbb{R}$ and $\pi_1(S_j)$ is the affine group Γ generated by the t_λ ($\lambda \in \Lambda$) and $\tilde{\sigma} : x \mapsto A_j x + v/m_j$, with $\tilde{\sigma} t_\lambda \tilde{\sigma}^{-1} = t_{A_j\lambda}$, $\tilde{\sigma}^{m_j} = t_v$ (Proposition 5.6) and $1 \to \Lambda \to \Gamma \to \mathbb{Z}/m_j \to 1$ exact. Abelianising, with $\Lambda_{A_j} := \Lambda/(A_j - I)\Lambda$,

$$H_1(S_j; \mathbb{Z}) = \big(\Lambda_{A_j} \oplus \mathbb{Z}\tilde{\sigma}\big)/\big\langle([v], -m_j)\big\rangle.$$

We claim (γ, ψ_j) induces $\Lambda_{A_j} \cong \mathbb{Z}^2$. Surjectivity is clear ($\hat{\gamma} \mapsto (1,0)$, and $\hat{w} \mapsto (0,1)$ for $j = 1$, $\hat{u} \mapsto (0,1)$ for $j = 2$). A direct computation gives generators of $(A_j - I)\Lambda$ and a basis k_1, k_2 of $K := \ker(\gamma, \psi_j)$:

$$j = 1: \quad 6\hat{u} - 6\hat{w} - 2\hat{\delta},\ -\hat{u} - \hat{w} + \hat{\delta},\ \hat{u} - 2\hat{w};$$
$$k_1 = \hat{u} - 2\hat{w}, \quad k_2 = -3\hat{w} + \hat{\delta},$$
$$j = 2: \quad -6\hat{w} + 3\hat{\delta},\ -\hat{u} + \hat{w},\ -\hat{u} - \hat{w} + \hat{\delta};$$
$$k_1 = \hat{u} - \hat{w}, \quad k_2 = -2\hat{w} + \hat{\delta}.$$

All listed generators lie in K. For $j = 1$ one has $k_2 = k_1 + (-\hat{u} - \hat{w} + \hat{\delta})$; for $j = 2$ one has

$$-\hat{u} + \hat{w} = -k_1, \qquad -\hat{u} - \hat{w} + \hat{\delta} = -k_1 + k_2, \qquad -6\hat{w} + 3\hat{\delta} = 3k_2.$$

Hence $(A_j - I)\Lambda = K$ in both cases. Therefore, for *every* admissible v,

$$H_1(S_j; \mathbb{Z}) \cong \mathbb{Z}^3/\big\langle(\gamma(v), \psi_j(v), -m_j)\big\rangle,$$

with $\psi_1(v) = 3c$, $\psi_2(v) = 2c$ in the notation of Proposition 5.6. Such a quotient is torsion free, and hence $\cong \mathbb{Z}^2$, precisely when the relation vector is primitive, i.e. when $\gcd(\gamma(v), \psi_j(v), m_j) = 1$; and this holds by admissibility, since $\gcd(\gamma(v), m_1) = \gcd(\gamma(v), 3) = 1$ when $3 \nmid \gamma(v)$, resp. $\gcd(\gamma(v), m_2) = \gcd(\gamma(v), 4) = 1$ when $\gamma(v)$ is odd. For the choices made for X, where $\psi_1(\varepsilon) = 0 = \psi_2(\varepsilon')$, this reads

$$H_1(S_1; \mathbb{Z}) = \mathbb{Z}^3/\langle(1, 0, -3)\rangle \cong \mathbb{Z}^2, \qquad H_1(S_2; \mathbb{Z}) = \mathbb{Z}^3/\langle(-1, 0, -4)\rangle \cong \mathbb{Z}^2.$$

The two flat bundles. The eigenvalues of A_j are $1, 1, \omega, \bar{\omega}$ with $\omega = e^{2\pi i/3}$ for $j = 1$, and $1, 1, i, -i$ for $j = 2$ (order m_j and $\operatorname{rk} \Lambda^{A_j} = 2$). Since $A_j \otimes \mathbb{C} \cong R_j \oplus \overline{R_j}$ (proof of Lemma 5.7) and the spectra $\{1, 1\}$ and $\{\omega, \bar{\omega}\}$ (resp. $\{i, -i\}$) for R_j are excluded, R_j has eigenvalues 1 and a primitive m_j-th root of unity, so $\det R_j$ is a primitive m_j-th root of unity. Trivialising $K_{A_0^{(j)}}$ by $\omega_0 = d\zeta_1 \wedge d\zeta_2$ and using $\sigma^*\omega_0 = \det(R_j)\omega_0$ gives $K_{S_j} \cong L_{\chi'}$, $\chi'(\sigma) = \det(R_j)^{-1}$, of order m_j by Lemma 5.7. The normal bundle of $A_0^{(j)}$ in J'_j is $A_0^{(j)} \times T_{z_j}\Delta_j$, on which $\tilde{g}_j^{\log}$ acts by $(x, \xi) \mapsto (\sigma x, \zeta_j \xi)$; hence $\mathcal{O}_{N_j}(S_j)|_{S_j} \cong N_{S_j/N_j} \cong L_\chi$ with $\chi(\sigma) = \zeta_j$, again of order exactly m_j. (Consistently, $\mathcal{O}(m_j S_j)|_{S_j} = f_j^*\mathcal{O}_{D_j}(p_j)|_{S_j}$ is trivial.)

Type. By Lemma 5.7, S_j is a free quotient of an abelian surface by a cyclic group whose generator has non-trivial linear part, hence bielliptic of Bagnera–de Franchis type [BdF], [BHPV, List V.5]: $S_j = (E \times F)/G$ with G one of $\mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/4, \mathbb{Z}/4 \times \mathbb{Z}/2, \mathbb{Z}/3, (\mathbb{Z}/3)^2, \mathbb{Z}/6$, for which $\operatorname{ord}(K_S) = 2, 2, 4, 4, 3, 3, 6$ and the torsion of $H_1(S; \mathbb{Z})$ is $(\mathbb{Z}/2)^2, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}/3, 0, 0$ respectively [BHPV, List V.5], [Ser]. Since $\operatorname{ord}(K_{S_1}) = 3$ and $\operatorname{ord}(K_{S_2}) = 4$ with H_1 torsion free, S_1 is of type $(\mathbb{Z}/3)^2$ and S_2 of type $\mathbb{Z}/4 \times \mathbb{Z}/2$. □

5.4 Regluing by the logarithmic section

Proof of Theorem 5.4(iv). σ_j is a section. On a simply connected $U \subset \Delta_j^*$ pick a branch ℓ of $\log s_j$ and set $\sigma_j^\ell(z) := \frac{\ell(z)}{2\pi i}\Pi(z)v$. Two branches differ by $2\pi i k$, $k \in \mathbb{Z}$, hence the maps differ by $k\Pi(z)v \in L(z)$ (as $v \in \Lambda$); the induced local sections of $J'_j|_{\Delta_j^*}$ therefore glue to a global holomorphic section σ_j, and the fibrewise translation $h_j[(z, \zeta)] = [(z, \zeta + \sigma_j^\ell(z))]$ is a well-defined biholomorphism of $J'_j|_{\Delta_j^*}$ over Δ_j^* (translations commute with the t_λ).

The conjugation identity. Fix $z_0 \in \Delta_j^*$, a simply connected $U \ni z_0$ and a branch ℓ of $\log s_j$ on U. Since $s_j(g_j z) = \zeta_j s_j(z)$ with $\zeta_j = e^{-2\pi i/m_j}$, the function $\ell'(w) := \ell(g_j^{-1}w) - \frac{2\pi i}{m_j}$ is a branch of $\log s_j$ on $g_j U$; we use ℓ on U and ℓ' on $g_j U$, which is legitimate as h_j is branch-independent. For $z \in U$, $\zeta \in \mathbb{C}^2$ we have

$$h_j^{-1}[(z, \zeta)] = \Big[\Big(z,\ \zeta - \tfrac{\ell(z)}{2\pi i}\Pi(z)v\Big)\Big],$$

and applying $\tilde{g}_j^{\log}$, then (5.2) and $A_j v = v$,

$$\begin{aligned}\tilde{g}_j^{\log} h_j^{-1}[(z, \zeta)] &= \Big[\Big(g_j z,\ R_{g_j}(z)\zeta - \tfrac{\ell(z)}{2\pi i}\Pi(g_j z)A_j v + \tfrac{1}{m_j}\Pi(g_j z)v\Big)\Big] \\ &= \Big[\Big(g_j z,\ R_{g_j}(z)\zeta - \Big(\tfrac{\ell(z)}{2\pi i} - \tfrac{1}{m_j}\Big)\Pi(g_j z)v\Big)\Big].\end{aligned}$$

Applying h_j at $g_j z \in g_j U$ adds

$$\frac{\ell'(g_j z)}{2\pi i}\Pi(g_j z)v = \Big(\frac{\ell(z)}{2\pi i} - \frac{1}{m_j}\Big)\Pi(g_j z)v,$$

so that

$$h_j\, \tilde{g}_j^{\log}\, h_j^{-1}[(z, \zeta)] = \tilde{g}_j^{0}[(z, \zeta)].$$

Both sides are globally defined on $J'_j|_{\Delta_j^*}$, so the identity holds globally.

Descent. Thus h_j maps $\langle \tilde{g}_j^{\log} \rangle$-orbits bijectively to $\langle \tilde{g}_j^{0} \rangle$-orbits; as both quotient maps are unramified coverings of complex manifolds, h_j descends to a biholomorphism G_j, whose target is $J|_{D_j^*}$ by (5.1). Both quotient maps commute with $s_j^{m_j} = t_j \circ \pi$ and h_j preserves fibres, so G_j commutes with the projections to D_j^*. □

Remark 5.8 (The twist parameter)*.* Theorem 5.4 holds for *every* $v \in \Lambda^{A_j}$ satisfying Proposition 5.6, with the *same* global family J: the surgery takes place over D_j only and G_j is always a biholomorphism onto $J|_{D_j^*}$. As $\Lambda^{A_1} = \langle \varepsilon, \hat{\delta} \rangle$, $\Lambda^{A_2} = \langle \varepsilon', \hat{\delta} \rangle$ with $\gamma(\varepsilon) = \gamma(\varepsilon') = 1$ and $\gamma(\hat{\delta}) = 0$, the integer $\gamma(v)$ realises every admissible residue class. By Lemma 5.2(iii), $\tilde{g}_j^{v/m_j} = \tilde{g}_j^{(v + m_j\mu)/m_j}$ for $\mu \in \Lambda^{A_j}$, so N_j depends on v only modulo $m_j\Lambda^{A_j}$, whereas G_j depends on v itself: replacing v by $v + m_j\mu$ composes G_j with translation by the single-valued section $\frac{\log t_j}{2\pi i}\Pi\mu$ of $J|_{D_j^*}$; this dependence is taken up in §6 and used in §7. For X we take $v_1 = \varepsilon$, $v_2 = -\varepsilon'$, so $(\ell_1, \ell_2) = (1, -1)$; for comparison, the threefold X' with $v_2 = +\varepsilon'$, i.e. $(\ell_1, \ell_2) = (1, 1)$, is equally admissible.

6 The compact complex manifold X.

Throughout we use the family $f_J\colon J \to B^\circ$ of §3, the toric filling $f_0\colon N_0 \to D_0$ of §4, the logarithmic fillings $f_j\colon N_j \to D_j$ $(j = 1, 2)$ of §5, and the gluing maps G_0, G_1, G_2 constructed there.

6.1 The glued space

Construction 6.1. Fix once and for all: (i) the orbifold cover $\pi\colon \mathfrak{h}_z \to B \setminus \{p_0\}$ with z_1, z_2, g_1, g_2 and the period functions τ, μ, β of §3, in particular a constant c_0 for which condition $(\beta 3)$ of Definition 3.1 holds; (ii) a radius $r > 0$ as in §3 (so μ and $\beta + \tau$ are bounded on U_r), a branch of $\log u_1$ on $\{|t_c| < r\}$, hence h, $s = \tau - h(t_c \circ \pi)$ and $C(t_c)$; (iii) a number ϵ with $0 < \epsilon < \min\{r, 1\}$ small enough that Theorem 4.5 applies, and linearising pairs $(\Delta_1, s_1), (\Delta_2, s_2)$ in the sense of §5, small enough that Theorem 5.4 applies, with branches of $\log s_j$ on simply connected subsets of Δ_j^*.

Put $D_0 := \{|t_c| < \epsilon\}$, $D_j := \pi(\Delta_j)$, $D_i^* := D_i \setminus \{p_i\}$; shrinking ϵ and Δ_j we assume D_0, D_1, D_2 pairwise disjoint, so that

$$B = B^\circ \cup D_0 \cup D_1 \cup D_2, \qquad D_i^* \subset B^\circ \ (i = 0, 1, 2). \tag{6.1}$$

By Proposition 4.7(iii) the map G_0 induced by E_0 is a biholomorphism $G_0\colon J|_{D_0^*} \xrightarrow{\sim} N_0 \setminus W$ with $f_0 \circ G_0 = f_J$, and by Theorem 5.4 the map G_j induced by h_j is a biholomorphism

$$G_j\colon N_j \setminus f_j^{-1}(p_j) \xrightarrow{\sim} J|_{D_j^*}, \qquad f_J \circ G_j = f_j \qquad (j = 1, 2).$$

Let $\mathcal{Z} := J \sqcup N_0 \sqcup N_1 \sqcup N_2$ and let $\sim$ be generated by $p \sim G_0(p)$ $(p \in J|_{D_0^*})$ and $q \sim G_j(q)$ $(q \in N_j \setminus f_j^{-1}(p_j))$. Set $X := \mathcal{Z}/\sim$ with the quotient topology, let $\iota_J, \iota_{N_0}, \iota_{N_1}, \iota_{N_2}$ be the induced maps, and let $f\colon X \to B$ equal f_J on $\iota_J(J)$, f_0 on $\iota_{N_0}(N_0)$ and f_j on $\iota_{N_j}(N_j)$.

Theorem 6.2. X is a connected compact Hausdorff complex manifold of dimension 3, and $f\colon X \to B \cong \mathbb{P}^1$ is a proper surjective holomorphic map with connected fibres. More precisely:

(1) $\iota_J, \iota_{N_0}, \iota_{N_1}, \iota_{N_2}$ are open holomorphic embeddings whose images cover X, with

$$f^{-1}(B^\circ) = \iota_J(J), \quad f^{-1}(D_0) = \iota_{N_0}(N_0), \quad f^{-1}(D_j) = \iota_{N_j}(N_j) \ (j = 1, 2),$$

and f corresponds under ι_J, resp. ι_{N_i}, to f_J, resp. f_i.

(2) For $b \in B^\circ$ the fibre $f^{-1}(b)$ is the complex 2-torus $\mathbb{C}^2/L(z)$, $\pi(z) = b$; $f^{-1}(p_0) = W$ as a reduced divisor; and $f^*(p_j) = m_j S_j$, $j = 1, 2$, with S_j the bielliptic surface of Theorem 5.4.

Proof. Since D_0, D_1, D_2 are pairwise disjoint, the sets $J|_{D_i^*}$ are pairwise disjoint in J, while the $N_i \setminus f_i^{-1}(p_i)$ lie in different summands; hence $\sim$ is already an equivalence relation whose nonsingleton classes are the pairs $\{p, G_0(p)\}$, $\{q, G_j(q)\}$, and each class meets each summand at most once. The saturation of an open $U \subset \mathcal{Z}$ is the union of U with the images of $U \cap J|_{D_i^*}$ and $U \cap (N_i \setminus f_i^{-1}(p_i))$ under $G_0^{\pm 1}, G_j^{\pm 1}$, hence open; so the quotient map is open and the four ι are open embeddings covering X. Transporting the complex structures gives an atlas whose only transition maps are the biholomorphisms $G_0^{\pm 1}, G_j^{\pm 1}$, so X is a complex 3-manifold. The three definitions of f agree on overlaps because $f_0 \circ G_0 = f_J$ and $f_J \circ G_j = f_j$; since $\iota_{N_0}(N_0 \setminus W)$ and $\iota_{N_j}(N_j \setminus f_j^{-1}(p_j))$ lie in $\iota_J(J)$, (1) follows, and (2) restates §3, Theorem 4.5 and Theorem 5.4.

Two points of X with distinct images in B are separated by f-preimages of disjoint discs, and two points with the same image lie, by (6.1) and (1), in a common Hausdorff chart $f^{-1}(V)$, $V \in \{B^\circ, D_0, D_1, D_2\}$; so X is Hausdorff. Over each such V the map f is f_J, f_0 or f_j, hence proper; properness being local on the base for maps of locally compact Hausdorff spaces, f is proper and $X = f^{-1}(B)$ compact. Finally J, N_0, N_j are connected, being quotients of $\mathfrak{h}_z^\circ \times \mathbb{C}^2$, of Y_ϵ and of $\mathcal{T}|_{\Delta_j}$, and each N_i meets $\iota_J(J)$; so X is connected, $f(X)$ is a compact analytic subset of B containing B°, i.e. f is surjective, and the fibres are connected by (2) with Theorems 4.5 and 5.4. $\square$

6.2 Independence of the auxiliary choices

Proposition 6.3. Let X and $\widetilde{X}$ be built by Construction 6.1 from the same π, z_j, g_j, the same τ, μ, β (in particular the same c_0) and the same v_1, v_2, but possibly from different r, ϵ, different branches of $\log u_1$ and $\log s_j$, and different linearising pairs (Δ_j, s_j). Then there is a biholomorphism $F\colon X \to \widetilde{X}$ with $\widetilde{f} \circ F = f$.

Proof. (a) Changing the branch of $\log u_1$ replaces h by $h + n$ ($n \in \mathbb{Z}$), hence s by $s - n$ and C by $C + nB_0$. For $\bar{\lambda} \in \overline{\Lambda} = \mathbb{Z}^2$ one has $nB_0\bar{\lambda} \in \mathbb{Z}^2$, so $c_{\bar{\lambda}} = \exp(2\pi i\, C\bar{\lambda})$, hence $\Psi_{\bar{\lambda}}$, N_0, f_0 are unchanged; and $e^{2\pi i(s-n)} = e^{2\pi i s}$, so E_0 and G_0 are unchanged.

(b) Changing the branch of $\log s_j$ replaces σ_j by $\sigma_j + k\,\Pi(z)v_j$, $k \in \mathbb{Z}$; as $v_j \in \Lambda$, translation by $\Pi(z)v_j$ is the identity on $\mathcal{T}|_{\Delta_j}$, so h_j and G_j are unchanged.

(c) *Shrinking.* Suppose $\widetilde{X}$ is built from $\tilde{r} \le r$, $\tilde{\epsilon} \le \epsilon$, $\widetilde{\Delta}_j \subset \Delta_j$, with the restricted branches and $\tilde{s}_j = s_j|_{\widetilde{\Delta}_j}$. Then $\widetilde{N}_0 \subset N_0$ and $\widetilde{N}_j \subset N_j$ are open with

$$\widetilde{G}_0 = G_0|_{J|_{\widetilde{D}_0^*}}, \qquad \widetilde{G}_j = G_j|_{\widetilde{N}_j \smallsetminus f_j^{-1}(p_j)},$$

so the identity on J together with these inclusions descends to a holomorphic injection $\widetilde{X} \to X$ over B; it is surjective because $\iota_{N_0}(N_0 \smallsetminus W) \subset \iota_J(J)$ and $\iota_{N_j}(N_j \smallsetminus f_j^{-1}(p_j)) \subset \iota_J(J)$, while W and S_j lie in the image. Hence it is a biholomorphism over B.

Reduction to a common choice. Recall (Definition 5.1): a *linearising pair* (Δ_j, s_j) consists of a g_j-invariant simply connected open $\Delta_j \ni z_j$ with $g\Delta_j \cap \Delta_j = \emptyset$ for $g \in \Delta \smallsetminus \langle g_j \rangle$, and an injective coordinate s_j with $s_j(z_j) = 0$, $s_j \circ g_j = \zeta_j s_j$, where

$$\zeta_j = e^{-2\pi i/m_j};$$

roundness of Δ_j in the coordinate s_j is *optional*, and a round sub-disc $\{|s_j| < \rho\}$ is always available inside Δ_j. In particular Δ_j is g_j-invariant and simply connected by definition; both facts will be used in (d). Every round sub-disc $\{|s_j| < \rho\}$ with the restricted coordinate is again a linearising pair in the sense of §5, so the shrinking step above applies to it.

Note next that N_j, f_j and D_j do not depend on the choice of the linearising coordinate on a given Δ_j (see (d)); only σ_j, h_j and G_j do. Thus for any domain D as above and any two linearising coordinates on it the two glued spaces are defined by the same formulas, and (d) compares them.

Now let (Δ_j, s_j) and (Δ_j', s_j') be two linearising pairs, and let Δ_j'' be the connected component of $\Delta_j \cap \Delta_j'$ containing z_j; it is an open g_j-invariant neighbourhood of z_j, because Δ_j, Δ_j' are g_j-invariant and g_j fixes z_j. Since $s_j(\Delta_j'')$ is an open neighbourhood of 0, choose $\rho > 0$ with $\{|s| < \rho\} \subset s_j(\Delta_j'')$ and put $D_\rho := s_j^{-1}\{|s| < \rho\} \subset \Delta_j''$, a g_j-invariant disc (s_j being injective with $s_j \circ g_j = \zeta_j s_j$); likewise choose $\rho' > 0$ with $\{|s| < \rho'\} \subset s_j'(D_\rho)$ and put $D'_{\rho'} := (s_j')^{-1}\{|s| < \rho'\} \subset D_\rho$. Both $(D_\rho, s_j|_{D_\rho})$ and $(D'_{\rho'}, s_j'|_{D'_{\rho'}})$ are round linearising pairs: they are simply connected and g_j-invariant, and $gD_\rho \cap D_\rho = \emptyset$ for $g \notin \langle g_j \rangle$ is inherited from Δ_j. By the shrinking step, X is biholomorphic over B to the space built from $(D_\rho, s_j|_{D_\rho})$, and $\widetilde{X}$ to the space built from $(D'_{\rho'}, s_j'|_{D'_{\rho'}})$. By (d) applied on D_ρ to the coordinates $s_j|_{D_\rho}$ and $s_j'|_{D_\rho}$, the space built from $(D_\rho, s_j|_{D_\rho})$ is biholomorphic over B to the one built from $(D_\rho, s_j'|_{D_\rho})$, and the latter is biholomorphic over B, again by shrinking (now for the coordinate s_j'), to the space built from $(D'_{\rho'}, s_j'|_{D'_{\rho'}})$. Likewise two choices of r, resp. of ϵ, admit a common smaller one, and branches are compared by (a) and (b). This proves the proposition, granted (d).

(d) Let s_j' be another linearising coordinate on a fixed Δ_j, i.e. $s_j' \circ g_j = e^{-2\pi i/m_j} s_j'$ and $s_j'(z_j) = 0$, $ds_j' \neq 0$. Being a coordinate, s_j' is injective on Δ_j, so it vanishes only at z_j and there to order 1, exactly as s_j does;

hence $U := s_j'/s_j$ extends holomorphically across z_j and is nowhere zero on Δ_j, and it is g_j-invariant because s_j, s_j' are multiplied by the same factor $e^{-2\pi i/m_j}$. (This is where injectivity is needed: for a non-injective substitution such as $s_j' = s_j^4$ the quotient would vanish at z_j and the comparison below would fail.) Since Δ_j is simply connected (part of Definition 5.1, recalled in (c)), the nowhere-zero holomorphic function U admits a holomorphic logarithm on Δ_j; fix one and put $\varphi := \log U/2\pi i$, a holomorphic function on Δ_j. The function $\varphi \circ g_j - \varphi$ is continuous and $\mathbb{Z}$-valued on the connected set Δ_j (because $U \circ g_j = U$), hence equals a constant $n \in \mathbb{Z}$; iterating m_j times and using $g_j^{m_j} = 1$ gives $m_j n = 0$, so $n = 0$ and

$$\varphi \circ g_j = \varphi. \tag{6.2}$$

(Equivalently: U descends to a nowhere vanishing holomorphic function of t_j on the disc D_j, which is simply connected, and φ is the pullback of a logarithm there.)

Neither $J_j' = \mathcal{T}|_{\Delta_j}$ nor $\tilde{g}_j^{\log}$ depends on s_j, so N_j and f_j are unchanged; only σ_j changes, by $\sigma_j' = \sigma_j + \psi$ with

$$\psi(z) := \varphi(z)\,\Pi(z)v_j,$$

a holomorphic $\mathbb{C}^2$-valued function on Δ_j. Let $T_\psi(z,\zeta) := (z, \zeta + \psi(z))$ on $\mathfrak{h}_z \times \mathbb{C}^2$ restricted to Δ_j. It commutes with the lattice translations $(z,\zeta) \mapsto (z, \zeta + \Pi(z)\lambda)$, $\lambda \in \Lambda$, and it commutes with $\tilde{g}_j^{\log}(z,\zeta) = (g_j z,\, R_{g_j}(z)\zeta + \Pi(g_j z)v_j/m_j)$: indeed, by (6.2) and $R_{g_j}(z)\Pi(z) = \Pi(g_j z)M_{g_j}^{-1} = \Pi(g_j z)A_j$ with $A_j v_j = v_j$,

$$R_{g_j}(z)\psi(z) = \varphi(z)\,\Pi(g_j z)A_j v_j = \varphi(g_j z)\,\Pi(g_j z)v_j = \psi(g_j z).$$

Hence T_ψ descends to an automorphism θ of N_j over D_j with $G_j' = G_j \circ \theta$. The identity on $J, N_0, N_{j'}$ $(j' \neq j)$ and θ^{-1} on N_j descends to a biholomorphism $X \to \widetilde{X}$ over B. $\square$

Remark 6.4. The construction does depend on c_0: already the fibres $\mathbb{C}^2/L(z)$ vary with it. We write X_{c_0} when the choice matters. The constructions of §§4–6 and the proofs of §§7–10 are written for an arbitrary admissible c_0 — admissible in the sense of condition $(\beta 3)$ of Definition 3.1 — and use the period functions τ, μ, β only through the laws and normalisations of §3, which hold for every admissible c_0. In particular no continuity or local constancy of X_{c_0} in the parameter c_0 enters the arguments. Whether $X_{c_0} \cong X_{c_0'}$ for admissible $c_0 \neq c_0'$ is left open.

6.3 The zero section

By Theorem 3.4 the maps $\tilde{g}$ are fibrewise linear, so $\tilde{g}(z,0) = (gz, 0)$ and $z \mapsto [(z,0)]$ descends to a holomorphic *zero section* $s_0\colon B^\circ \to J \subset X$, $f \circ s_0 = \mathrm{id}$. Its behaviour at Σ is the input for §7.

Lemma 6.5.

(i) s_0 extends holomorphically across p_0 to a section of f over $B^\circ \cup \{p_0\}$, with value at p_0 the image of the point $(1,1)$ of the open orbit of the toric divisor $D_{(0,0)}$, a point of $W \setminus \mathrm{D}$; the extension meets W transversally there.

(ii) For $j = 1, 2$ the section $G_j^{-1} \circ s_0$ of f_j over D_j^* is the image in N_j of the holomorphic map $z \mapsto [(z, \varsigma_j(z))]$ on $J_j'|_{\Delta_j^*}$, where

$$\varsigma_j(z) := -\sigma_j(z) = -\frac{\log s_j(z)}{2\pi i}\,\Pi(z)v_j.$$

It is independent of the branch of $\log s_j$ and is $\tilde{g}_j^{\log}$-invariant. Following it along $s_j = \varrho e^{i\theta}$, $\theta \in [0, 2\pi]$ *(*during which $t_j = s_j^{m_j}$ traverses $|t_j| = \varrho^{m_j}$ exactly m_j times counterclockwise*)*, the holomorphic lift ends at $\varsigma_j(z) - \Pi(z)v_j$; in the marking $H_1(\text{fibre}) = \Lambda$ of §3 the section drifts by $-v_j$.

(iii) s_0 does not extend to a holomorphic section of f over any neighbourhood of p_j, $j = 1, 2$.

Proof. (i) As G_0 is induced by $E_0(z,\zeta) = (e^{2\pi i\zeta_1}, e^{2\pi i\zeta_2}, e^{2\pi i s(z)})$, the image $G_0 \circ s_0$ is that of the curve $\{((1,1),t_c) : 0 < |t_c| < \epsilon\} \subset T_{N'} \cap Y_\epsilon$. Let $\sigma \in \mathfrak{F}$ be the cone over $\operatorname{conv}\{(0,0), e_1, e_2\} \times \{1\}$; its dual is generated by the basis $\mathsf{m}_1 = (1,0,0)$, $\mathsf{m}_2 = (0,1,0)$, $\mathsf{m}_0 = (-1,-1,1)$ of M', so $U_\sigma \cong \mathbb{C}^3$ with coordinates $(\chi^{\mathsf{m}_0}, \chi^{\mathsf{m}_1}, \chi^{\mathsf{m}_2}) = (t/x_1x_2,\, x_1,\, x_2)$, and $\{\chi^{\mathsf{m}_0} = 0\}$ is the toric divisor $D_{(0,0)}$ of the ray through $(0,0,1)$. In these coordinates the curve is $\{(t_c,1,1) : 0 < |t_c| < \epsilon\}$, whose closure is the holomorphic disc

$$\mathbb{D}_{\mathrm{sec}} := \{(t_c,1,1) : |t_c| < \epsilon\} \subset Y_\epsilon,$$

meeting $D_{(0,0)}$ transversally at $(0,1,1)$, a point of its open orbit; t restricts to t_c on $\mathbb{D}_{\mathrm{sec}}$.

Let $q\colon Y_\epsilon \to N_0$ be the quotient map. By Theorem 4.5, q is a local biholomorphism, and it is moreover *injective on* $\mathbb{D}_{\mathrm{sec}}$: the deck transformations $\Psi_{\bar\lambda}$ preserve the function t (Lemma 4.3(iii), which gives $t \circ \Psi_{\bar\lambda} = t$ and hence $f_0 \circ q = t$), while $t|_{\mathbb{D}_{\mathrm{sec}}} = t_c$ is injective, so two distinct points of $\mathbb{D}_{\mathrm{sec}}$ cannot be identified by q. A local biholomorphism which is injective on $\mathbb{D}_{\mathrm{sec}}$ maps $\mathbb{D}_{\mathrm{sec}}$ biholomorphically onto its image, preserving transversality of intersections; since $f_0 \circ q|_{\mathbb{D}_{\mathrm{sec}}} = t_c$ is a biholomorphism onto D_0, the image $q(\mathbb{D}_{\mathrm{sec}})$ is a holomorphic section of f_0 over D_0 extending $G_0 \circ s_0$, meeting W transversally in the point $q(0,1,1)$, which lies in the open orbit of $D_{(0,0)}$ and hence in $W \smallsetminus \mathsf{D}$.

(ii) Changing the branch replaces ς_j by $\varsigma_j - k\Pi(z)v_j$, $k \in \mathbb{Z}$, $v_j \in \Lambda$, so the section is unchanged. From $\zeta_j = e^{-2\pi i/m_j}$, i.e. $\log s_j \circ g_j = \log s_j - 2\pi i/m_j$, together with $R_{g_j}(z)\Pi(z) = \Pi(g_jz)M_{g_j}^{-1} = \Pi(g_jz)A_j$ and $A_jv_j = v_j$,

$$R_{g_j}(z)\varsigma_j(z) + \frac{\Pi(g_jz)v_j}{m_j} = -\frac{\log s_j(z)}{2\pi i}\,\Pi(g_jz)v_j + \frac{\Pi(g_jz)v_j}{m_j} = \varsigma_j(g_jz),$$

which is $\tilde g_j^{\log}$-invariance. As h_j is translation by $+\sigma_j$, it carries $[(z,\varsigma_j(z))]$ to $[(z,0)]$; the drift follows since $\log s_j$ increases by $2\pi i$ along the loop, so that ς_j changes by $-\Pi(z)v_j$, the period of the class $-v_j \in \Lambda$.

(iii) If ς were such a section, then after shrinking it is a section of f_j over D_j with $t_j|_\varsigma$ a coordinate, so $(f_j \circ \varsigma)^*(p_j)$ is the reduced point $\varsigma(p_j)$; but $f_j^*(p_j) = m_jS_j$ and $\varsigma \not\subset S_j$, whence this divisor is $m_j(S_j \cdot \varsigma) \geq m_j\,\varsigma(p_j)$, contradicting $m_j \geq 3$. □

6.4 The discrete gluing parameter ℓ_0

The gluings at p_1, p_2 involved the choices $v_j \in \Lambda^{A_j}$, recorded by $\ell_j = \gamma(v_j)$; the gluing at p_0 was tautological (Proposition 4.7(iii)), and the freedom thereby given up is named as follows. Let $\mathcal{J}$ be the sheaf on D_0^* of germs of holomorphic sections of f_J, a sheaf of abelian groups with origin s_0. By Theorem 3.4, $R_{g_0} \equiv I$ and $\Pi(s+1) = \Pi(s)M_0$, so in the coordinate s of §4

$$\begin{aligned} J|_{D_0^*} &= \big(\{\operatorname{Im} s > h_0\} \times \mathbb{C}^2\big)/\big(\Lambda \rtimes \langle g\rangle\big), \\ g(s,\zeta) &= (s+1,\zeta), \qquad \lambda(s,\zeta) = (s,\zeta + \Pi(s)\lambda). \end{aligned} \tag{6.3}$$

Definition 6.6. For $\sigma \in H^0(D_0^*, \mathcal{J})$ lift σ to a holomorphic $z\colon \{\operatorname{Im} s > h_0\} \to \mathbb{C}^2$ (the half-plane being simply connected); by (6.3) there is a unique $\lambda_\sigma \in \Lambda$ with $z(s+1) - z(s) = \Pi(s+1)\lambda_\sigma$. Replacing z by $z + \Pi(s)\lambda_0$ replaces λ_σ by $\lambda_\sigma + (I - M_0^{-1})\lambda_0 \in \lambda_\sigma + \Lambda_{\mathrm{tor}}$, since $M_0^{-1} - I = N^t$ has image in Λ_{tor}. Hence

$$c\colon H^0(D_0^*, \mathcal{J}) \longrightarrow \overline{\Lambda} = \Lambda/\Lambda_{\mathrm{tor}}, \qquad c(\sigma) := [\lambda_\sigma],$$

is a well-defined homomorphism; it is the connecting map of the exponential sequence $0 \to \mathbf{\Lambda} \to \mathcal{O}^2_{D_0^*} \to \mathcal{J} \to 0$ under $H^1(D_0^*, \mathbf{\Lambda}) \cong \Lambda/(M_0 - I)\Lambda = \overline{\Lambda}$, $\mathbf{\Lambda}$ being the local system of periods [Hat02].

Proposition 6.7.

(i) c is surjective.

(ii) For $\sigma \in H^0(D_0^*, \mathcal{J})$ let T_σ be the fibrewise translation by σ and $X^{(\sigma)}$ the space obtained from Construction 6.1 with G_0 replaced by $G_0 \circ T_\sigma$. Then $X^{(\sigma)}$ satisfies Theorem 6.2 as well, and with $\ell_0(X^{(\sigma)}) := \gamma(c(\sigma)) \in \mathbb{Z}$ (γ kills Λ_{tor}, hence descends to $\overline{\Lambda}$) every integer occurs.

(iii) The space X of Construction 6.1 is $X^{(0)}$ and has $\ell_0 = 0$.

Proof. (i) With $Z(s) = sB_0 + C(t_c)$, $t_c = e^{2\pi i s}$, one has $\Pi(s)\hat{u} = (s+h, \mu)^t$ and $\Pi(s)\hat{\gamma} = (6\mu, b-s-h)^t$, where h, μ, b are holomorphic in t_c on $\{|t_c| < r\}$, hence invariant under $s \mapsto s+1$ (§3). A direct computation gives

$$z_{\hat{u}}(s) := \left(\tfrac{s^2+s}{2} + s\,h,\ s\,\mu\right), \qquad z_{\hat{\gamma}}(s) := \left(6s\,\mu,\ -\tfrac{s^2+s}{2} + s\,(b-h)\right),$$
$$z_{\hat{u}}(s+1) - z_{\hat{u}}(s) = \Pi(s+1)\hat{u}, \qquad z_{\hat{\gamma}}(s+1) - z_{\hat{\gamma}}(s) = \Pi(s+1)\hat{\gamma};$$

so these define $\sigma_{\hat{u}}, \sigma_{\hat{\gamma}}$ with $c(\sigma_{\hat{u}}) = \bar{\hat{u}}$, $c(\sigma_{\hat{\gamma}}) = \bar{\hat{\gamma}}$, which generate $\overline{\Lambda}$. (Alternatively D_0^* is Stein, so $H^1(D_0^*, \mathcal{O}^2) = 0$ by Cartan's Theorem B [GrRe].)

(ii) T_σ is a biholomorphism over D_0^*, so $G_0 \circ T_\sigma$ is again a biholomorphism $J|_{D_0^*} \to N_0 \smallsetminus W$ over D_0^* and the proof of Theorem 6.2 applies verbatim; since $\gamma(\hat{\gamma}) = 1$, (i) gives every integer value. (iii) is immediate from $c(0) = 0$. □

Remark 6.8. Thus ℓ_0 is a discrete parameter of the p_0-gluing with values in $\gamma(\overline{\Lambda}) = \mathbb{Z}$, and it is not a function of the transcendental constants $\mu(p_0), b(0), h(0)$: the filling N_0 is manufactured from $C(t_c)$ itself, and twists with $c(\sigma) \in \Lambda_{\text{tor}}$ leave ℓ_0 unchanged. The triple $(\ell_0, \ell_1, \ell_2) = (0, \gamma(v_1), \gamma(v_2)) = (0, 1, -1)$ is the complete list of discrete gluing data of X, and enters $\pi_1(X)$ in §7.

7 Topology of X: fundamental group and integral homology.

7.1 Decomposition, collapse, and the pieces

The integral homology of X is computed twice on the following pages, by two independent routes: first by the *collapse/Mayer–Vietoris* computation — the collapse of a general fibre onto the central fibre W (Proposition 7.2, proved in §7.2), the homology of W and the specialisation maps (§7.3), and Mayer–Vietoris for $X = N_0' \cup X^\circ$ (§7.6), culminating in Theorem 7.22; and secondly by the *Leray* computation of §7.7 ("A second computation") (for admissible pairs with $4 \mid \varepsilon_2(v_2)$, in particular for X), which recovers H^1, H^2, H^3 of X from $E_2^{a,b} = H^a(B, R^b f_*\mathbb{Z})$. Neither computation uses the other: the Leray route takes as input only the invariants statement at p_0 (Theorem B.1, proved in Appendix B by nearby cycles, without §7.2), the local data at p_1, p_2 of §7.4 (Lemma 7.13, Proposition 7.14), the meridian identifications of §7.5 (with Lemma 7.16) and Corollary 4.8; it uses neither Proposition 7.2 nor Proposition 7.11 nor §7.6 (Theorem 7.22); see Corollary 7.29.

Throughout, (co)homology has coefficients $\mathbb{Z}$ and F denotes a general fibre of f, a real 4-torus; by §2

$$H_1(F) = \Lambda, \qquad H^1(F) = V,$$
$$H_q(F) = \textstyle\bigwedge^q \Lambda, \qquad H^q(F) = \textstyle\bigwedge^q V,$$

in the bases $(\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta})$, (γ, u, w, δ). All meridians occurring in this section are the *clockwise* ones of Convention 3.17. The monodromy along the (clockwise) meridian a_j of p_j is T_j on H^1 and A_j on H_1 $(j = 1, 2)$, along $a_0 := (a_1 a_2)^{-1}$ it is T_0, resp. M_0 (§3), and on H^q it is $\bigwedge^q T_j$, resp. $\bigwedge^q T_0$.

Notation 7.1. Fix $0 < \eta \ll 1$ and put $N_0' := f^{-1}\{|t_c| \le \eta\}$, $N_j' := f^{-1}\{|t_j| \le \eta\}$ $(j = 1, 2)$, and

$$X^\circ := \overline{X \setminus N_0'} = f^{-1}(D'), \qquad D' := B \setminus \{|t_c| < \eta\}$$

a closed disc containing p_1, p_2. Write $M := \partial N_0'$, $M_j := \partial N_j'$. The letter X' is reserved for the comparison threefold of §6, built from the same data with $v_2 = +\varepsilon'$.

By §4 and §5 the maps $f_0 : N_0 \to \Delta_\varepsilon$ and $f_j : N_j \to D_j$ are proper holomorphic, $f_0^{-1}(0) = W$ is a reduced normal-crossings divisor ($t = z_1 z_2 z_3$ in suitable local coordinates at each point) and $f_j^{-1}(p_j) = m_j S_j$ with S_j smooth. Here W is irreducible, hence normal crossings but not simple normal crossings, whereas the retraction and collapse statements of Clemens [Cle77] and Persson [Per77] are proved for semistable degenerations with *simple* normal-crossings central fibre; we quote them for context. The statement used below is the following one, read off the toric model of §4 and proved in §7.2.

Proposition 7.2. There is $\eta_0 > 0$ with the following property. Let $0 < \eta \le \eta_0$ and fix t_0 with $0 < |t_0| \le \eta$; put $F := f^{-1}(t_0)$. Then there is a strong deformation retraction $r : N_0' \to W$, induced by a $\{\Psi_{\bar\lambda}\}$- and T_f-equivariant strong deformation retraction of $\{|t| \le \eta\} \subset Y_\varepsilon$ onto the central fibre $Y_0 = t^{-1}(0)$, such that the restriction

$$c := r|_F : F \longrightarrow W$$

is described as follows. Write

$$F = T_B \times T_f, \qquad T_B = (\overline{\Lambda} \otimes \mathbb{R})/\overline{\Lambda}, \quad T_f = (\Lambda_{\mathrm{tor}} \otimes \mathbb{R})/\Lambda_{\mathrm{tor}},$$

for the splitting induced by the toric chart (T_f the compact torus of $(\mathbb{C}^*)^2$, T_B the Log-torus carrying the induced A_2-triangulation), where throughout we identify $T_B = (\overline{\Lambda} \otimes \mathbb{R})/\overline{\Lambda} \cong (\Lambda_{\mathrm{tor}} \otimes \mathbb{R})/\Lambda_{\mathrm{tor}}$ by means of the unimodular shear $B_0 : \overline{\Lambda} \xrightarrow{\sim} \Lambda_{\mathrm{tor}}$ of §2 (the form in which T_B appears in the proof, §7.2, Lemma 7.10), and let $\Theta \subset T_B$ be the dual graph of that triangulation: two vertices (the two triangles of $\mathcal{T}_{A_2}/\mathbb{Z}^2$) and three edges e_1, e_2, e_3, where e_k is dual to the triangulation edge of direction $n_k \in \Lambda_{\mathrm{tor}}$,

$$n_1 = \hat{w}, \qquad n_2 = \hat{\delta} - \hat{w}, \qquad n_3 = -\hat{\delta},$$

so that $n_1 + n_2 + n_3 = 0$ and any two of the n_k form a basis of Λ_{tor}. Then c collapses T_f to a point over each vertex of Θ (producing the triple points P, Q of W), collapses $S^1_{n_k} \subset T_f$ over the interior of e_k (with image the double curve $\bar{C}_k \cong \mathbb{P}^1$ of D), and is injective over $T_B \setminus \Theta$. For any other t with $0 < |t| \le \eta$, the restriction $r|_{f^{-1}(t)}$ is homotopic, as a map to W, to a map with the same description.

Lemma 7.3. For η small:

(i) $W \hookrightarrow N_0'$ and $S_j \hookrightarrow N_j'$ are deformation retracts, so $H_*(N_0') \cong H_*(W)$ and $H_*(N_j') \cong H_*(S_j)$;

(ii) $X = N_0' \cup X^\circ$ with $N_0' \cap X^\circ = M$ (collars making the pieces open).

Proof. (ii) is clear. For (i) at p_0 the retraction is the one constructed in Proposition 7.2 (§7.2; the proof uses only §4); for N_j none is needed, since smoothly $N_j = (A_0^{(j)} \times \Delta_{s_j})/\langle \tilde{g}_j^{\log} \rangle$ and the radial retraction of Δ_{s_j} is equivariant. □

Lemma 7.4. For η small:

(i) f is a locally trivial bundle over $\{0 < |t_c| \le \eta\}$ and over $\{0 < |t_j| \le \eta\}$; the geometric monodromies are affine with linear parts M_0, resp. A_j, hence isotopic to M_0, resp. A_j, so that M is diffeomorphic to the mapping torus of M_0 on F and M_j to that of A_j.

Moreover $\pi_1(N_0) \cong \overline{\Lambda} \cong \mathbb{Z}^2$ and

$$\pi_1(M) = \Lambda \rtimes_{M_0} \mathbb{Z}\langle \tilde{a}_0 \rangle, \qquad \tilde{a}_0 t_\lambda \tilde{a}_0^{-1} = t_{M_0\lambda},$$

where $\tilde{a}_0$ is the class of the circle $\{x = x_0\} \subset Y_\varepsilon$ over $|t| = \eta$, oriented *clockwise*, i.e. by $s \mapsto s - 1$ for $s = \frac{\log t}{2\pi i}$ (the orientation of Convention 3.17, and the one for which the displayed conjugation rule $\tilde{a}_0 t_\lambda \tilde{a}_0^{-1} = t_{M_0\lambda}$ holds); we call it the *toric meridian*. Furthermore $\ker(\pi_1(M) \to \pi_1(N_0))$ is the normal closure of $\Lambda_{\mathrm{tor}} \cup \{\tilde{a}_0\}$; for $j = 1, 2$

$$\pi_1(N_j) \cong \pi_1(S_j) = \langle \Lambda, \hat{g}_j \mid \hat{g}_j \lambda \hat{g}_j^{-1} = A_j \lambda,\ \hat{g}_j^{m_j} = t_{v_j} \rangle,$$

$\pi_1(M_j) = \Lambda \rtimes_{A_j} \mathbb{Z}\langle \hat{g}_j \rangle$, and $\ker(\pi_1(M_j) \to \pi_1(N_j))$ is the normal closure of the meridian $\hat{g}_j^{m_j} t_{v_j}^{-1}$.

Proof. (i) is Ehresmann's theorem with the monodromies of §3; an affine diffeomorphism of a torus is isotopic to its linear part through the straight-line path on the translation part.

By Corollary 4.8, Y_ε is simply connected; as $N_0 = Y_\varepsilon/\{\Psi_{\bar\lambda}\}$ with a free properly discontinuous $\overline{\Lambda}$-action, $\pi_1(N_0) = \overline{\Lambda}$. Over $0 < |t| < \varepsilon$ the covering is $((\mathbb{C}^*)^2 \times \Delta^*)/\overline{\Lambda}$, with fundamental group the extension of $\overline{\Lambda}$ by $\Lambda_{\mathrm{tor}} \oplus \mathbb{Z}\tilde{a}_0$; the map to $\pi_1(N_0)$ kills exactly Λ_{tor} and $\tilde{a}_0$. The conjugation rule for $\tilde{a}_0$ is the monodromy of §3 in the (clockwise) orientation of $a_0 = (a_1 a_2)^{-1}$, since $M_0 = (A_1 A_2)^{-1}$. For N_j the deck group of $A_0^{(j)} \times \Delta_{s_j} \to N_j$ is $\langle \tilde{g}_j^{\log} \rangle \cong \mathbb{Z}/m_j$, acting freely, and $S_j = A_0^{(j)}/\langle x \mapsto A_j x + v_j/m_j \rangle$ (§5), the affine map having m_j-th power t_{v_j} because $v_j \in \Lambda^{A_j}$.

For the kernel at p_j we compare presentations; since the fibre is multiple, N_j' is not a tubular neighbourhood of S_j, and the meridian statement for tubular neighbourhoods does not apply. Let $U := A_0^{(j)} \times \Delta_{s_j} \to N_j$ be the $\mathbb{Z}/m_j$-covering with deck group generated by

$$\tilde{g}_j^{\log} : (x, z) \mapsto (A_j x + v_j/m_j,\ \zeta z), \qquad \zeta = e^{-2\pi i/m_j},$$

the disc factor being forced by the normalisation $s_j \circ g_j = \zeta_j s_j$ of §5, and let $U_\partial := A_0^{(j)} \times \partial\Delta_{s'} \to M_j$ be its restriction over a circle $|z| = s'$, again a $\mathbb{Z}/m_j$-covering with the same deck group. Then $\pi_1(U) = \Lambda$, $\pi_1(U_\partial) = \Lambda \times \mathbb{Z}\langle\sigma\rangle$ with σ the class of $\{\mathrm{pt}\} \times \partial\Delta_{s'}$, oriented counterclockwise in the z-disc, and $\pi_1(U_\partial) \to \pi_1(U)$ is the quotient by the central subgroup $\langle\sigma\rangle$ (the circle bounds the disc $\Delta_{s'}$). Since the inclusion $U_\partial \hookrightarrow U$ is equivariant for the same deck group, the induced map of extensions

$$1 \to \pi_1(U_\partial) \to \pi_1(M_j) \to \mathbb{Z}/m_j \to 1, \qquad 1 \to \pi_1(U) \to \pi_1(N_j) \to \mathbb{Z}/m_j \to 1$$

shows that $\pi_1(M_j) \to \pi_1(N_j)$ is onto with kernel the normal closure of σ. Finally, $\hat{g}_j$ is a lift to $\pi_1(M_j)$ of the deck generator $\tilde{g}_j^{\log}$: it is represented by a path in U_∂ from (x_0, z_0) to $\tilde{g}_j^{\log}(x_0, z_0)$, along which the z-coordinate turns by $-2\pi/m_j$ (the disc factor being $\zeta = e^{-2\pi i/m_j}$) and the $A_0^{(j)}$-coordinate moves by v_j/m_j. Hence $\hat{g}_j^{m_j}$ is represented by the loop travelling once around $\partial\Delta_{s'}$ *negatively* while the $A_0^{(j)}$-coordinate drifts by $\big(\sum_{i<m_j} A_j^i\big) v_j/m_j = v_j$ (using $A_j^{m_j} = I$ and $v_j \in \Lambda^{A_j}$); as σ is central in $\pi_1(U_\partial)$ this reads

$$\hat{g}_j^{m_j} = t_{v_j}\sigma^{-1}, \qquad \text{i.e.} \qquad \sigma^{-1} = \hat{g}_j^{m_j} t_{v_j}^{-1},$$

whose normal closure is that of σ, i.e. the asserted generator of the kernel. Killing σ turns the semidirect-product presentation of $\pi_1(M_j)$ into the displayed presentation of $\pi_1(S_j)$, with $\hat{g}_j^{m_j} = t_{v_j}$. □

7.2 Proof of Proposition 7.2

We use the toric model of §4 throughout: $Y = Y_{\mathfrak{F}}$ with torus $T_{N'} = (\mathbb{C}^*)^3$, coordinates (x_1, x_2, t), compact torus

$$T_c := N' \otimes S^1 = (S^1)^3, \qquad T_f := \Lambda_{\mathrm{tor}} \otimes S^1 = \{(*, *, 1)\} \subset T_c,$$

$Y_{\le\eta} := \{|t| \le \eta\}$, $Y_0 = t^{-1}(0) = \bigcup_{v \in \mathbb{Z}^2} E_v$ with $E_v \cong \mathrm{dP}_6$, and $\Gamma := \{\Psi_{\bar\lambda}\}_{\bar\lambda \in \overline{\Lambda}}$, which by Theorem 4.5 acts freely and properly discontinuously on Y_ε, preserves Y_0 and every $\{|t| \le \eta\}$, and has quotients $N_0' = Y_{\le\eta}/\Gamma$, $W = Y_0/\Gamma$. Note that $Y \setminus Y_0 = T_{N'}$, every orbit of dimension < 3 lying in some E_v. For a torus point $p = (x, t)$ we write $y(p) = \log|x| / \log|t| \in \mathbb{R}^2$ for its position (§4).

The proof is broken into six lemmas (Lemmas 7.5–7.10), assembled at the end of the subsection.

Notation used throughout the subsection. For $0 < |t| < \varepsilon$ put $s := \frac{\log t}{2\pi i}$ and $Z_t := sB_0 + C(t)$, so that on the fibre $Y_t = (\mathbb{C}^*)^2$, in logarithmic coordinates $\xi := \frac{1}{2\pi i}\log x \in \mathbb{C}^2/\mathbb{Z}^2$, the map $\Psi_{\bar\lambda}$ is the translation $\xi \mapsto \xi + Z_t\bar\lambda$ (§4). The branch of $\log t$ is immaterial for everything that follows: another branch replaces s by $s + n$ and Z_t by $Z_t + nB_0$ ($n \in \mathbb{Z}$), hence the translation vectors $Z_t\bar\lambda$ by $Z_t\bar\lambda + nB_0\bar\lambda \in Z_t\bar\lambda + \mathbb{Z}^2$ ($B_0 : \overline{\Lambda} \to \Lambda_{\mathrm{tor}}$ being integral), which is the same translation of $\mathbb{C}^2/\mathbb{Z}^2$; and $\operatorname{Im} Z_t$, B_t and the map L_t below are unchanged outright. Since $\operatorname{Im}\xi = -\frac{\log|t|}{2\pi}\, y$, setting

$$\mathsf{R}(t) := -2\pi \operatorname{Im} C(t), \qquad B_t := B_0 + \mathsf{R}(t)/\log|t|$$

we get $\operatorname{Im} Z_t = -\frac{\log|t|}{2\pi} B_t$. Write $C_0 := C(0)$, $Z_t^0 := sB_0 + C_0$, $B_t^0 := B_0 + \mathsf{R}(0)/\log|t|$, $c_{\bar\lambda}^0 := \exp(2\pi i C_0 \bar\lambda)$ and

$$\Psi_{\bar\lambda}^0 := (c_{\bar\lambda}^0, 1) \cdot \Phi_{\bar\lambda}, \qquad \Gamma^0 := \{\Psi_{\bar\lambda}^0\}_{\bar\lambda \in \overline{\Lambda}}$$

(the action with the twist frozen at $t = 0$; on Y_0 it coincides with Γ, because $c_{\bar\lambda}(0) = c_{\bar\lambda}^0$).

Lemma 7.5 (Straightening to a constant twist)**.** There is η_0 with $0 < \eta_0 < \min(\varepsilon, 1)$ — so that $Y_{\le\eta_0} \subset Y_\varepsilon$ and $\log|t| < 0$ for $0 < |t| \le \eta_0$ — such that $\|B_t - B_0\| \le \frac{1}{2}$, hence $B_t \in GL_2(\mathbb{R})$, and likewise $B_t^0 \in GL_2(\mathbb{R})$, for all $0 < |t| \le \eta_0$. Fix $0 < \eta \le \eta_0$ and define

$$G(x, t) := \Big(\exp\big(-2\pi i\,(C(t) - C_0)\,B_t^{-1} y(x, t)\big) \cdot x,\ t\Big) \quad (t \ne 0), \qquad G|_{Y_0} := \mathrm{id}.$$

Then G is a homeomorphism of $Y_{\leq\eta}$ over $\{|t| \leq \eta\}$ with the properties:

(a) $G|_{Y_0} = \mathrm{id}$; G preserves t and commutes with T_f;

(b) $G \circ \Psi_{\bar\lambda} = \Psi^0_{\bar\lambda} \circ G$ for every $\bar\lambda \in \overline{\Lambda}$;

(c) on each fibre Y_t $(0 < |t| \leq \eta)$, G induces a real-linear isomorphism $\bar G_t : Y_t/\Gamma \to Y_t/\Gamma^0$ of the quotient 4-tori which is the identity on $H_1 = \Lambda$ with respect to the markings $\Pi_t = [Z_t \mid I]$ and $\Pi^0_t = [Z^0_t \mid I]$ (it fixes e_1, e_2 and carries $Z_t\bar\lambda$ to $Z^0_t\bar\lambda$); here Γ, Γ^0 preserve Y_t since $\Psi_{\bar\lambda}$ preserves t, and (b) makes $\bar G_t$ well defined.

Consequently, if R is a Γ^0- and T_c-equivariant strong deformation retraction of $Y_{\leq\eta}$ onto Y_0, then $G^{-1}R_sG$ is a Γ- and T_f-equivariant strong deformation retraction of $Y_{\leq\eta}$ onto Y_0, and it descends to a strong deformation retraction of $N'_0 = Y_{\leq\eta}/\Gamma$ onto $W = Y_0/\Gamma$.

Proof. By the boundedness of C on $\{|t_c| \leq r/2\}$ (§4, the bound $\|\mathsf{R}\|_\infty$ of Theorem 4.5) we may fix $\eta_0 < \min(\varepsilon, 1)$ so small that $\|B_t - B_0\| \leq \frac{1}{2}$ for $0 < |t| \leq \eta_0$; then $B_t \in GL_2(\mathbb{R})$, and the same estimate and conclusion hold for $B^0_t = B_0 + \mathsf{R}(0)/\log|t|$. Fix $0 < \eta \leq \eta_0$.

In ξ-coordinates $G|_{Y_t}$ is induced by the $\mathbb{R}$-linear map

$$L_t(\xi) = \xi - (C(t) - C_0)(\operatorname{Im} Z_t)^{-1} \operatorname{Im}\xi,$$

which is the unique $\mathbb{R}$-linear map with $L_t|_{\mathbb{R}^2} = \mathrm{id}$ and $L_t(Z_t\bar\lambda) = Z^0_t\bar\lambda$ for all $\bar\lambda \in \mathbb{R}^2$: writing $\xi = Z_ta + b$ with $a = (\operatorname{Im} Z_t)^{-1}\operatorname{Im}\xi$ and $b \in \mathbb{R}^2$, one has $L_t\xi = Z^0_ta + b$. Consequently:

(i) L_t carries the $\mathbb{R}$-basis $(e_1, e_2, Z_te_1, Z_te_2)$ of $\mathbb{C}^2$ to the $\mathbb{R}$-basis $(e_1, e_2, Z^0_te_1, Z^0_te_2)$, hence is invertible; being the identity on $\mathbb{R}^2 \supset \mathbb{Z}^2$ it descends to a diffeomorphism of $\mathbb{C}^2/\mathbb{Z}^2 = Y_t$.

(ii) $G \circ \Psi_{\bar\lambda} = \Psi^0_{\bar\lambda} \circ G$ on $Y_{\leq\eta} \setminus Y_0$, because L_t is linear and carries the translation vector $Z_t\bar\lambda$ to $Z^0_t\bar\lambda$; on Y_0 the two actions agree and $G = \mathrm{id}$. This is (b).

(iii) G preserves t and commutes with T_f, since it multiplies x by a torus element depending only on (y, t) and T_f does not change y; together with $G|_{Y_0} = \mathrm{id}$ this is (a).

(iv) G and G^{-1} (given by the same formula with $(C(t) - C_0)(B^0_t)^{-1}$ in place of $-(C(t) - C_0)B_t^{-1}$) are continuous on $Y_{\leq\eta}$. Away from Y_0 this is clear. At $p_* \in Y_0$ continuity is a local statement, so we may first shrink: choose a triangle σ with $p_* \in \mathcal{U}_\sigma$ and, by Lemma 4.4(i), a shrunken subchart $\mathcal{U}_\sigma(S) \ni p_*$ on which the positions $y(p)$ of torus points $p \in \mathcal{U}_\sigma(S) \cap Y_{\leq\eta}$ remain in a fixed bounded set (on all of $\mathcal{U}_\sigma \cong \mathbb{C}^3$ the positions are unbounded), while $C(t) - C_0 \to 0$ and B_t^{-1} stays bounded as $t \to 0$; hence the acting torus element tends to 1 and $G(p) \to p_* = G(p_*)$ for $p \to p_*$ inside $\mathcal{U}_\sigma(S)$. Points of Y_0 near p_* are fixed.

So G is a homeomorphism of $Y_{\leq\eta}$ over $\{|t| \leq \eta\}$ with (a), (b); and on each fibre it is induced by the real-linear automorphism L_t, whence a real-linear isomorphism $\bar G_t : Y_t/\Gamma \to Y_t/\Gamma^0$ of the quotient 4-tori which is the identity on $H_1 = \Lambda$ with respect to the markings $\Pi_t = [Z_t \mid I]$ and $\Pi^0_t = [Z^0_t \mid I]$ (it fixes e_1, e_2 and carries $Z_t\bar\lambda$ to $Z^0_t\bar\lambda$); here Γ, Γ^0 preserve Y_t since $\Psi_{\bar\lambda}$ preserves t, and (b) makes $\bar G_t$ well defined. This is (c).

Finally, let R be a Γ^0- and T_c-equivariant strong deformation retraction of $Y_{\leq\eta}$ onto Y_0. Then $G^{-1}R_sG$ is a strong deformation retraction of $Y_{\leq\eta}$ onto Y_0 (as G is a homeomorphism fixing Y_0 pointwise), it is Γ-equivariant by (b) and T_f-equivariant by (a); being Γ-equivariant it descends to the quotients $N'_0 = Y_{\leq\eta}/\Gamma$ and $W = Y_0/\Gamma$. □

Lemma 7.6 (Polar decomposition)**.** Let $Y_{\geq 0} \subset Y$ be the closure of $(\mathbb{R}_{>0})^3 \subset T_{N'}$ and $(Y_{\leq\eta})_{\geq 0} := Y_{\geq 0} \cap Y_{\leq\eta}$. Then:

(i) the multiplication map $m : T_c \times Y_{\geq 0} \to Y$, $(\phi, q) \mapsto \phi \cdot q$, is continuous, surjective and closed, hence a quotient map; the same holds for its restriction $T_c \times (Y_{\leq\eta})_{\geq 0} \to Y_{\leq\eta}$;

(ii) $\phi \cdot q = \phi' \cdot q'$ iff $q = q'$ and $\phi^{-1}\phi' \in \operatorname{Stab}_{T_c}(q)$; the *modulus* $p \mapsto |p| \in Y_{\geq 0}$ is a continuous retraction;

(iii) if q lies in the orbit $O_{\hat{C}}$ of the cone $\hat{C}$ over a cell C of $\mathcal{T}_{A_2}$, then $\mathrm{Stab}_{T_c}(q) = T_{\hat{C}}$, the compact subtorus attached to $N_{\hat{C}} := N' \cap \mathrm{span}_{\mathbb{R}} \hat{C} = \bigoplus_{v \in V(C)} \mathbb{Z}(v,1)$ (for the orbit O_σ of a cone σ the stabiliser in $T_{N'}$ of each of its points is the subtorus on $N_\sigma := N' \cap \mathrm{span}_{\mathbb{R}} \sigma$ [Ful93, §3.1]; its intersection with $T_c = N' \otimes S^1$ is $N_\sigma \otimes S^1$, connected because N_σ is saturated in N'), and

$$T_{\hat{C}} \cap T_f = \begin{cases} 1, & C = \{v\} \text{ a vertex}, \\ S^1_{v'-v}, & C = \{v, v'\} \text{ an edge}, \\ T_f, & C \text{ a triangle}. \end{cases} \tag{7.1}$$

Proof. (i) In the chart $\mathcal{U}_\sigma \cong \mathbb{C}^3$ of a triangle σ (coordinates z_0, z_1, z_2, characters positive on $(\mathbb{R}_{>0})^3$), $Y_{\geq 0}$ is the orthant $[0,\infty)^3$ and $z = \phi \cdot |z|$ with $|z| = (|z_0|, |z_1|, |z_2|)$; so m is continuous and surjective. It is closed, T_c being compact and Y locally compact Hausdorff; hence it is a quotient map, and likewise after restriction over $\{|t| \leq \eta\}$.

(ii) Again chartwise clear, as is the continuity of the modulus.

(iii) $N_{\hat{C}}$ equals $\bigoplus_{v \in V(C)} \mathbb{Z}\,(v,1)$ because $\mathcal{T}_{A_2}$ is unimodular [Ful93, §3.1]. An element $\sum_v a_v(v,1)$ of its Lie algebra lies in $\Lambda_{\mathrm{tor}} \otimes \mathbb{R}$ iff $\sum_v a_v = 0$, whence (7.1). The Lie-algebra computation identifies a priori only the identity component, and the intersection is connected here. Indeed, both $N_{\hat{C}}$ and Λ_{tor} are saturated in N', and $N_{\hat{C}} + \Lambda_{\mathrm{tor}} = N'$ in each of the three cases (any $(v,1) \in \hat{C}$ together with $\Lambda_{\mathrm{tor}} = \mathbb{Z}^2 \times \{0\}$ generates N'), so in particular $N' \cap (\mathrm{span}_{\mathbb{R}} \hat{C} + \Lambda_{\mathrm{tor}} \otimes \mathbb{R}) = N'$ is saturated; and for two subtori $T_{L_1}, T_{L_2} \subset N' \otimes S^1$ attached to saturated sublattices with $L_1 + L_2$ saturated one has $T_{L_1} \cap T_{L_2} = T_{L_1 \cap L_2}$, which is connected. (If $a \in L_1 \otimes \mathbb{R}$ is congruent mod N' to $b \in L_2 \otimes \mathbb{R}$, then $a - b \in N' \cap ((L_1 + L_2) \otimes \mathbb{R}) = L_1 + L_2$, say $a - b = n_1 + n_2$ with $n_i \in L_i$; then $a - n_1 = b + n_2 \in (L_1 \cap L_2) \otimes \mathbb{R}$, so a represents a point of $T_{L_1 \cap L_2}$.) $\square$

Lemma 7.7 (The positive part and the honeycomb)**.** Polar-decompose the constant twist: $c^0_{\bar{\lambda}} = u_{\bar{\lambda}} a_{\bar{\lambda}}$ with $u_{\bar{\lambda}} = \exp(2\pi i \,\mathrm{Re}(C_0 \bar{\lambda})) \in T_f$ and $a_{\bar{\lambda}} = |c^0_{\bar{\lambda}}| = \exp(\mathsf{R}_0 \bar{\lambda}) \in (\mathbb{R}_{>0})^2$, $\mathsf{R}_0 := \mathsf{R}(0)$, and put

$$\Psi^+_{\bar{\lambda}} := (a_{\bar{\lambda}}, 1) \cdot \Phi_{\bar{\lambda}}, \qquad \Gamma^+ := \{\Psi^+_{\bar{\lambda}}\}.$$

Let $0 < \eta \leq \eta_0$. Then:

(i) for $\phi \in T_c$, $q \in Y_{\geq 0}$,

$$\Psi^0_{\bar{\lambda}}(\phi \cdot q) = \big(u_{\bar{\lambda}} \varphi_{\bar{\lambda}}(\phi)\big) \cdot \big(\Psi^+_{\bar{\lambda}} q\big), \qquad \varphi_{\bar{\lambda}}(\phi) := \Phi_{\bar{\lambda}} \phi \Phi^{-1}_{\bar{\lambda}} \in T_c, \tag{7.2}$$

where, in the coordinates $(\phi, \theta) \in T_f \times S^1$ of T_c,

$$\varphi_{\bar{\lambda}}(\phi, \theta) = (\phi\, \theta^{B_0 \bar{\lambda}}, \theta); \tag{7.3}$$

in particular $\varphi_{\bar{\lambda}}|_{T_f} = \mathrm{id}$ (the level $\theta = 1$);

(ii) Γ^+ preserves $Y_{\geq 0}$ and acts freely and properly discontinuously on Y_ε; the quotient Y_ε / Γ^+ is the complex manifold N_0 of Theorem 4.5 for the constant twist $C^+ := (2\pi i)^{-1} \mathsf{R}_0$ (same R), and the induced map $Y_\varepsilon / \Gamma^+ \to \Delta_\varepsilon$ is proper (Theorem 4.5(c)); in particular $Y_{\leq \eta} / \Gamma^+$ is compact for $\eta < \varepsilon$. It acts on the positive slice $S_{t_1} := \{|t| = t_1\} \cap Y_{\geq 0} \cong \mathbb{R}^2_y$ $(0 < t_1 \leq \eta)$ by the translations $y \mapsto y + B^+_{t_1} \bar{\lambda}$, $B^+_{t_1} := B_0 + \mathsf{R}_0 / \log t_1$, and on Y_0 cellularly, $\Psi^+_{\bar{\lambda}}(E_v)_{\geq 0} = (E_{v + B_0 \bar{\lambda}})_{\geq 0}$;

(iii) $(Y_{\leq \eta})_{\geq 0}$ is a contractible topological 3-manifold with boundary $(Y_0)_{\geq 0} \sqcup \{|t| = \eta\}_{>0}$, with interior $\{0 < |t| < \eta\}_{>0} \cong \mathbb{R}^2_y \times (0, \eta)$ via $p \mapsto (y(p), |t(p)|)$;

(iv) there is a Γ^+-equivariant homeomorphism

$$\vartheta : \mathbb{R}^2 \longrightarrow (Y_0)_{\geq 0} \qquad (\text{source action } y \mapsto y + B_0 \bar{\lambda})$$

with $\vartheta(H_v) = (E_v)_{\geq 0}$, $\vartheta(H_v \cap H_{v'}) = (E_v \cap E_{v'})_{\geq 0}$ and $\vartheta(\text{vertex dual to } \sigma) = O_{\hat{\sigma}}$, where $\mathcal{H}$ is the honeycomb dual to $\mathcal{T}_{A_2}$ (hexagon H_v centred at $v \in \mathbb{Z}^2$); in particular $(Y_0)_{\geq 0} \cong \mathbb{R}^2$ is contractible.

Proof. (i) Since $\Phi_{\bar{\lambda}}$ is a toric automorphism it normalises T_c, whence (7.2); $\varphi_{\bar{\lambda}}$ is induced by the unipotent lattice map $(y, y_3) \mapsto (y + y_3 B_0 \bar{\lambda}, y_3)$, which is (7.3).

(ii) Thus Γ^+ preserves $Y_{\geq 0}$; it is the action of Theorem 4.5 for the constant twist $C^+ := (2\pi i)^{-1}\mathsf{R}_0$ (which has the same R at $t=0$, and $\|\mathsf{R}_0\| \leq \|\mathsf{R}\|_\infty$, so that the hypotheses of that Theorem hold on Y_ε), hence free and properly discontinuous by Theorem 4.5(b), the quotient Y_ε/Γ^+ being the complex manifold N_0 of that Theorem for the twist C^+, with proper induced map $Y_\varepsilon/\Gamma^+ \to \Delta_\varepsilon$ by Theorem 4.5(c); in particular $Y_{\leq\eta}/\Gamma^+$ is compact for $\eta < \varepsilon$. The formulas on S_{t_1} and on Y_0 are those of Lemma 4.3.

(iii) In a chart $(Y_{\leq\eta})_{\geq 0}$ is $\{r \in [0,\infty)^3 : r_0r_1r_2 \leq \eta\}$; near a triple point all r_i are small and the local model is $[0,\infty)^3 \cong \mathbb{R}^2 \times [0,\infty)$; near an interior point of a double arc exactly two of the r_i vanish, and the local model is $[0,\infty)^2 \times (0,\infty) \cong \mathbb{R}^2 \times [0,\infty)$; near an interior point of some $(E_v)_{\geq 0}$ exactly one r_i vanishes, giving $[0,\infty) \times (0,\infty)^2$; and near $\{|t| = \eta\}$ all $r_i > 0$ and $d(r_0r_1r_2) \neq 0$. Its boundary is $(Y_0)_{\geq 0} \sqcup \{|t| = \eta\}_{>0}$ and its interior is $\{0 < |t| < \eta\}_{>0} \cong \mathbb{R}^2_y \times (0,\eta)$ via $p \mapsto (y(p), |t(p)|)$. By the collar theorem [Bro62] the inclusion $\mathrm{int}(Y_{\leq\eta})_{\geq 0} \hookrightarrow (Y_{\leq\eta})_{\geq 0}$ is a homotopy equivalence, so $(Y_{\leq\eta})_{\geq 0}$ is contractible.

(iv) First, $(Y_0)_{\geq 0} = \bigcup_v (E_v)_{\geq 0}$, and $(E_v)_{\geq 0}$, the non-negative real part of the smooth projective toric surface $E_v \cong \mathrm{dP}_6$, is homeomorphic to its moment hexagon [Ful93, §4.1]: the moment map of E_v restricts to a homeomorphism of $(E_v)_{\geq 0}$ onto the moment polygon, carrying the non-negative part of each orbit closure onto the corresponding face; concretely, a closed disc whose boundary is a cycle of six arcs $(E_v \cap E_{v'})_{\geq 0} \cong [0,\infty]$, v' running through the six neighbours of v, consecutive arcs meeting at the T_c-fixed points $O_{\hat\sigma}$, $\sigma \ni v$. By Lemma 4.2, $(E_v)_{\geq 0} \cap (E_{v'})_{\geq 0} = (E_v \cap E_{v'})_{\geq 0}$ when $v' - v$ is an edge direction of $\mathcal{T}_{A_2}$ and is empty otherwise. Thus the closed cells $(E_v)_{\geq 0}$, $(E_v \cap E_{v'})_{\geq 0}$, $O_{\hat\sigma}$ have exactly the incidences of the hexagons, edges and vertices of the honeycomb $\mathcal{H}$ dual to $\mathcal{T}_{A_2}$ (hexagon H_v centred at $v \in \mathbb{Z}^2$, edge $H_v \cap H_{v'}$ dual to the triangulation edge $\{v, v'\}$, vertex dual to the triangle σ).

We construct ϑ by skeleta of $\mathcal{H}$, each step being made on representatives of the (free) Γ^+-action on cells and propagated equivariantly.

Vertices. Γ^+ permutes the vertices of $\mathcal{H}$ freely, with two orbits, dual to the two triangles of $\mathcal{T}_{A_2}/\Lambda_{\mathrm{tor}}$. The assignment $\vartheta(\text{vertex dual to } \sigma) := O_{\hat\sigma}$ is forced, and it is equivariant because $\Psi^+_{\bar\lambda} O_{\hat\sigma} = O_{\widehat{\sigma + B_0\bar\lambda}}$ (Lemma 4.3).

Edges. Γ^+ permutes the edges of $\mathcal{H}$ freely, with three orbits, represented by the three edges $A_{v'}$ of $\partial\overline{H_0}$ dual to $v' \in \{e_1, e_2, e_1 - e_2\}$. On each $A_{v'}$ choose a homeomorphism onto the arc $(E_0 \cap E_{v'})_{\geq 0} \cong [0,1]$ restricting at the two endpoints of $A_{v'}$ to the two values already fixed (possible, those endpoints being dual to the two triangles containing the edge $\{0, v'\}$, and their prescribed images the two T_c-fixed points of that arc), and propagate by

$$\vartheta|_{\Psi^+_{\bar\lambda} A_{v'}} := \Psi^+_{\bar\lambda} \circ \vartheta|_{A_{v'}} \circ (\,\cdot - B_0\bar\lambda).$$

In particular the three remaining edges of $\partial\overline{H_0}$, namely $A_{-v'} = A_{v'} - v'$, receive $\vartheta|_{A_{-v'}} = (\Psi^+_{\bar\lambda})^{-1} \circ \vartheta|_{A_{v'}} \circ (\,\cdot + v')$ where $B_0\bar\lambda = v'$ (note $(\Psi^+_{\bar\lambda})^{-1}(E_0 \cap E_{v'}) = E_{-v'} \cap E_0$). Since the vertex map of the previous step is equivariant, each of the six edge maps so obtained takes at each of its two endpoints exactly the prescribed vertex value; hence they agree at the six vertices of $\partial\overline{H_0}$ and assemble into a homeomorphism $\partial\overline{H_0} \to \partial(E_0)_{\geq 0}$ of circles.

2-*cells.* Γ^+ permutes the closed hexagons freely and transitively (B_0 being unimodular), with representative $\overline{H_0}$. The above homeomorphism of circles extends to a homeomorphism $\vartheta|_{\overline{H_0}} : \overline{H_0} \to (E_0)_{\geq 0}$ of the closed discs (Schoenflies/Alexander); put $\vartheta|_{\overline{H_v}} := \Psi^+_{\bar\lambda} \circ \vartheta|_{\overline{H_0}} \circ (\,\cdot - v)$ for $B_0\bar\lambda = v$. On an edge shared by two closed hexagons the two resulting definitions agree, because each of them is, by construction, the equivariant propagation of the map chosen on the corresponding edge representative $A_{v'}$. Since the closed hexagons form a locally finite closed cover of $\mathbb{R}^2$ and the map is continuous on each of them, the resulting map ϑ is well defined and continuous. It is bijective, because it carries the open hexagons, the open edges and the vertices bijectively onto the pairwise disjoint strata $(O_{\hat v})_{>0}$, $(O_{\widehat{\{v,v'\}}})_{>0}$, $O_{\hat\sigma}$, whose union is $(Y_0)_{\geq 0}$. It is proper: a compact $K \subset (Y_0)_{\geq 0}$ meets only finitely many $(E_v)_{\geq 0}$ — the family $\{E_v\}$ is locally finite in Y, since $E_v \subset \bigcup_{\sigma \ni v} \mathcal{U}_\sigma$ and each chart $\mathcal{U}_\sigma$ meets only the three E_v attached to the vertices of σ — so $\vartheta^{-1}(K)$ is a closed subset of a finite union of compact hexagons, hence compact. A continuous proper bijection between locally compact Hausdorff spaces is a homeomorphism; thus ϑ is a Γ^+-equivariant homeomorphism. In particular $(Y_0)_{\geq 0} \cong \mathbb{R}^2$ is contractible. ◻

Lemma 7.8 (Retracting the positive quotient)**.** Let $0 < \eta \le \eta_0$ and put

$$Q^+ := (Y_{\le\eta})_{\ge 0}/\Gamma^+, \qquad \mathsf{H} := (Y_0)_{\ge 0}/\Gamma^+ \subset Q^+.$$

Then Q^+ is a compact topological 3-manifold with boundary $\mathsf{H} \sqcup T^2_\eta$, where $T^2_\eta := \{|t| = \eta\}_{\ge 0}/\Gamma^+ \cong \mathbb{R}^2_y/B^+_\eta\overline{\Lambda}$ is a 2-torus (Γ^+ acts on the slice by $y \mapsto y + B^+_\eta\bar{\lambda}$, Lemma 7.7(ii), and $B^+_\eta \in GL_2(\mathbb{R})$, Lemma 7.5), the boundary component H is a 2-torus, and there exists

$$\bar{P} : Q^+ \times [0,1] \to Q^+, \qquad \bar{P}_0 = \mathrm{id}, \quad \bar{P}_s|_{\mathsf{H}} = \mathrm{id}, \quad \bar{P}_1(Q^+) = \mathsf{H}.$$

Proof. The action of Γ^+ on $(Y_{\le\eta})_{\ge 0}$ is free and properly discontinuous, being the restriction to the closed invariant subset $(Y_{\le\eta})_{\ge 0} \subset Y_\varepsilon$ of the free properly discontinuous action of Theorem 4.5 for the data (B_0, C^+) (Lemma 7.7(ii)); and it preserves the boundary $(Y_0)_{\ge 0} \sqcup \{|t| = \eta\}_{>0}$ of the manifold-with-boundary $(Y_{\le\eta})_{\ge 0}$ (Lemma 7.7(iii)), since it preserves Y_0 and the function $|t|$. Hence Q^+ is a topological 3-manifold with boundary $\mathsf{H} \sqcup T^2_\eta$. It is compact: $A := (Y_{\le\eta})_{\ge 0}$ is closed and Γ^+-invariant, i.e. saturated for the open quotient map $q : Y_{\le\eta} \to Y_{\le\eta}/\Gamma^+$; hence $q(A)$ is closed (its complement is the image of the invariant open set $Y_{\le\eta} \setminus A$), so compact, $Y_{\le\eta}/\Gamma^+$ being compact (Lemma 7.7(ii)); and the restriction of a quotient map to a saturated closed subset is a quotient map onto its image, so $Q^+ = A/\Gamma^+ \cong q(A)$. Its universal cover $(Y_{\le\eta})_{\ge 0}$ is contractible (Lemma 7.7(iii)); and $\mathsf{H} \cong \mathbb{R}^2/B_0\overline{\Lambda}$ (via ϑ, Lemma 7.7(iv)) is a 2-torus with contractible universal cover $(Y_0)_{\ge 0}$. Thus Q^+ and H are both $K(\mathbb{Z}^2, 1)$'s, and the inclusion $\mathsf{H} \hookrightarrow Q^+$, being covered by the Γ^+-equivariant inclusion of the two contractible covers, induces the identity of Γ^+ on π_1 and isomorphisms on all π_k. Compact manifolds are ANRs, hence of CW homotopy type, so $\mathsf{H} \hookrightarrow Q^+$ is a homotopy equivalence (Whitehead, [Hat02, Thm. 4.5]). As H is a boundary component of the compact manifold-with-boundary Q^+ it has a collar [Bro62], so the inclusion is a cofibration, and a homotopy equivalence which is a cofibration is the inclusion of a strong deformation retract [Hat02, Prop. 0.19, Cor. 0.20]: such a $\bar{P}$ exists. $\square$

Lemma 7.9 (Spreading by phases)**.** Let $0 < \eta \le \eta_0$ and let $P : (Y_{\le\eta})_{\ge 0} \times [0,1] \to (Y_{\le\eta})_{\ge 0}$ be continuous with

(P1) $P_0 = \mathrm{id}$;

(P2) $P_s|_{(Y_0)_{\ge 0}} = \mathrm{id}$ for all s;

(P3) $P_1\big((Y_{\le\eta})_{\ge 0}\big) \subset (Y_0)_{\ge 0}$;

(P4) every P_s is Γ^+-equivariant.

Then

$$R_s(\phi \cdot q) := \phi \cdot P_s(q) \qquad (\phi \in T_c,\ q \in (Y_{\le\eta})_{\ge 0})$$

is a well-defined, continuous, Γ^0- and T_c-equivariant strong deformation retraction of $Y_{\le\eta}$ onto Y_0; consequently (Lemma 7.5) $G^{-1}R_sG$ is a Γ- and T_f-equivariant strong deformation retraction of $Y_{\le\eta}$ onto Y_0, which descends to a strong deformation retraction r_s of N_0' onto W. Moreover such a P exists: the lift of the retraction $\bar{P}$ of Lemma 7.8 through the universal covering $(Y_{\le\eta})_{\ge 0} \to Q^+$ with $P_0 = \mathrm{id}$ satisfies (P1)–(P4).

Proof. Existence. Lift $\bar{P}$ through the universal covering $(Y_{\le\eta})_{\ge 0} \to Q^+$ to the unique homotopy P_s with $P_0 = \mathrm{id}$. By uniqueness of lifts, P_s is Γ^+-equivariant, $P_s|_{(Y_0)_{\ge 0}} = \mathrm{id}$, and P_1 is a retraction onto $(Y_0)_{\ge 0}$; i.e. (P1)–(P4) hold.

The construction. Let now P satisfy (P1)–(P4). Then R_s is well defined: if $\phi \cdot q = \phi' \cdot q$ then $\phi^{-1}\phi' \in \mathrm{Stab}_{T_c}(q)$ (Lemma 7.6(ii)), and either q lies in the interior, where the stabiliser is trivial, or $q \in \partial$; on $\{|t| = \eta\}_{>0}$ the stabiliser is again trivial, and on $(Y_0)_{\ge 0}$ we have $P_s(q) = q$ by (P2), so $\phi \cdot P_s(q) = \phi' \cdot P_s(q)$. It is continuous because $R_s \circ m = m \circ (\mathrm{id} \times P_s)$ and $m \times \mathrm{id}_{[0,1]}$ is a quotient map (m is closed by Lemma 7.6(i) and $[0,1]$ is compact; the product of a closed quotient map with the identity of a locally compact space is again a quotient map). By construction R commutes with T_c, and by (7.2)

$$R_s\big(\Psi^0_{\bar{\lambda}}(\phi \cdot q)\big) = R_s\big(u_{\bar{\lambda}}\varphi_{\bar{\lambda}}(\phi) \cdot \Psi^+_{\bar{\lambda}} q\big) = u_{\bar{\lambda}}\varphi_{\bar{\lambda}}(\phi) \cdot \Psi^+_{\bar{\lambda}} P_s(q) = \Psi^0_{\bar{\lambda}}\big(R_s(\phi \cdot q)\big),$$

using (P4). Finally $R_0 = \mathrm{id}$ by (P1), $R_s|_{Y_0} = \mathrm{id}$ by (P2) (as $Y_0 = T_c \cdot (Y_0)_{\geq 0}$) and $R_1(Y_{\leq \eta}) \subset T_c \cdot (Y_0)_{\geq 0} = Y_0$ by (P3). So R is a Γ^0- and T_c-equivariant strong deformation retraction of $Y_{\leq \eta}$ onto Y_0, and the last assertion is the final clause of Lemma 7.5. $\square$

Lemma 7.10 (The collapse on the fixed fibre)**.** Let $0 < \eta \leq \eta_0$, fix t_0 with $0 < |t_0| \leq \eta$ and put

$$\rho_0 := |t_0|, \qquad S := S_{\rho_0}, \qquad Y_{t_0} := \{t = t_0\}, \qquad F := f^{-1}(t_0) = \pi(Y_{t_0}),$$

$\pi : Y_{\leq \eta} \to N_0' = Y_{\leq \eta}/\Gamma$ being the quotient map. Let $\ell_{t_0} : \mathbb{R}^2 \to \mathbb{R}^2$, $y \mapsto B_0(B^+_{\rho_0})^{-1} y$, and let $\tilde{c} : Y_{t_0} \to Y_0$, $c_\vartheta : F^0_{t_0} := Y_{t_0}/\Gamma^0 \to W$ be as defined in the proof below. Let P satisfy (P1)–(P4) of Lemma 7.9 and let r_s be the resulting strong deformation retraction of N_0' onto W. Then:

(i) there is a splitting $F^0_{t_0} \cong T_B \times T_f$, $T_B := S/\Gamma^+$, under which c_ϑ collapses T_f to a triple point of W over each vertex of $\Theta = \mathcal{H}/B_0\overline{\Lambda}$, collapses $S^1_{n_k}$ over the interior of e_k onto the double curve $\bar{C}_k \cong \mathbb{P}^1$, and is injective over $T_B \setminus \Theta$, where it is onto $W \setminus \mathrm{D}$;

(ii) there is a homotopy P' again satisfying (P1)–(P4) and with $P_1'|_S = \vartheta \circ \ell_{t_0} \circ y$; for the corresponding strong deformation retraction r_s' of N_0' onto W furnished by Lemma 7.9 and for $r := r_1'$ one has

$$r|_F = c_\vartheta \circ \bar{G},$$

where $\bar{G} : F \to F^0_{t_0}$, $\bar{G}(\pi p) = \pi^0(Gp)$, is the homeomorphism of tori induced by G, the identity on the Λ-markings;

(iii) for every t with $0 < |t| \leq \eta$, the same construction carried out at the level $|t|$ gives

$$r|_{f^{-1}(t)} \simeq c^{(t)}_\vartheta \circ \bar{G}_t$$

as maps to W, where $\bar{G}_t : f^{-1}(t) \to F^0_t := Y_t/\Gamma^0$ is the level-$|t|$ analogue of $\bar{G}$, with $c^{(t)}_\vartheta : F^0_t \to W$ the map of the same shape as in (i), built at level $|t|$.

Proof. Write elements of T_c as (ϕ, θ), $\phi \in T_f$, $\theta \in S^1$, and put $\theta_0 := e^{i \arg t_0}$. Then

$$Y_{t_0} = \{t = t_0\} = (T_f, \arg t_0) \cdot S \cong T_f \times \mathbb{R}^2_y,$$

the stabilisers of points of S being trivial. The map $\ell_{t_0} : y \mapsto \tilde{y} := B_0(B^+_{|t_0|})^{-1} y$ is a linear isomorphism intertwining the Γ^+-action $y \mapsto y + B^+_{|t_0|}\bar{\lambda}$ on S with the action $\tilde{y} \mapsto \tilde{y} + B_0\bar{\lambda}$ on the source of ϑ. Define

$$\tilde{c} : Y_{t_0} \to Y_0, \qquad \tilde{c}\big((\phi, \arg t_0) \cdot q\big) := (\phi, \arg t_0) \cdot \vartheta\big(\ell_{t_0}(y(q))\big) \qquad (\phi \in T_f,\ q \in S).$$

As in Lemma 7.9 this is well defined — every point of S (indeed of $(Y_{\leq \eta})_{>0}$) has trivial T_c-stabiliser, so no compatibility condition arises — and continuous (by the quotient map m), T_f-equivariant, and Γ^0-equivariant (by the one-line computation of Lemma 7.9 with P_s replaced by the Γ^+-equivariant map $\vartheta \circ \ell_{t_0} \circ y$, using (7.2)); it therefore descends to

$$c_\vartheta : F^0_{t_0} = Y_{t_0}/\Gamma^0 \longrightarrow W.$$

Cell description; proof of (i). By Lemma 7.6, two points (ϕ, y), (ϕ', y) of $Y_{t_0} \cong T_f \times \mathbb{R}^2$ have the same image iff $\phi^{-1}\phi' \in \mathrm{Stab}_{T_c}(\vartheta(\tilde{y})) \cap T_f$, which is given by (7.1). Hence: over the open hexagon H_v° the map $\tilde{c}$ is injective, with image $(T_f, \arg t_0) \cdot (O_{\hat{v}})_{>0}$, and this is all of the open orbit $O_{\hat{v}} \cong (\mathbb{C}^*)^2$ of E_v: every point of $O_{\hat{v}}$ has stabiliser $T_{\hat{v}} = S^1_{(v,1)}$ and $T_c = T_f \cdot T_{\hat{v}}$ (because $\Lambda_{\mathrm{tor}} + \mathbb{Z}(v, 1) = N'$), so $O_{\hat{v}} = T_c \cdot (O_{\hat{v}})_{>0} = T_f \cdot (O_{\hat{v}})_{>0}$. Over the interior of the honeycomb edge dual to the triangulation edge $\{v, v'\}$ exactly the circle $S^1_{v'-v} \subset T_f$ is collapsed, the image being the 1-dimensional orbit whose closure is the curve $E_v \cap E_{v'} \cong \mathbb{P}^1$; over a honeycomb vertex all of T_f is collapsed to the triple point $O_{\hat{\sigma}}$.

Passing to the quotient by Γ^0: the slice torus is $T_B := S/\Gamma^+ \cong \mathbb{R}^2_{\tilde{y}}/B_0\overline{\Lambda} = (\Lambda_{\mathrm{tor}} \otimes \mathbb{R})/\Lambda_{\mathrm{tor}}$, on which the honeycomb descends to the graph $\Theta = \mathcal{H}/B_0\overline{\Lambda}$ with two vertices (the two triangles of $\mathcal{T}_{A_2}/\mathbb{Z}^2$) and three edges e_1, e_2, e_3, the edge e_k being dual to the triangulation edge of direction n_k, i.e. $n_1 = \hat{w}$, $n_2 = \hat{\delta} - \hat{w}$, $n_3 = -\hat{\delta}$ with $\hat{w} \leftrightarrow e_1$, $\hat{\delta} \leftrightarrow e_2$. On the T_f-factor Γ^0 acts by phases only: by (7.2) and (7.3), $\Psi^0_{\bar{\lambda}}$ multiplies the T_c-coordinate by $u_{\bar{\lambda}}\varphi_{\bar{\lambda}}(\phi, \theta)$, and since all points of Y_{t_0} are written $(\phi, \arg t_0) \cdot q$, the relevant level is $\theta = \theta_0$, at which the T_f-component of the action is multiplication by

$$u_{\bar\lambda}\,\theta_0^{B_0\bar\lambda} = \exp\big(2\pi i\,K_{t_0}\bar\lambda\big), \qquad K_{t_0} := \mathrm{Re}(C_0) + \tfrac{\arg t_0}{2\pi} B_0$$

($\varphi_{\bar\lambda}|_{T_f} = \mathrm{id}$ only at $\theta = 1$; the element $\theta_0^{B_0\bar\lambda}$ is well defined because $B_0\bar\lambda \in \Lambda_{\mathrm{tor}}$ is integral, so the choice of $\arg t_0$ is immaterial). Thus the Γ^0-action on $Y_{t_0} \cong T_f \times \mathbb{R}^2_y$ is

$$\bar\lambda \cdot (\phi, y) = \big(\phi\, e^{2\pi i K_{t_0}\bar\lambda},\ y + B^+_{|t_0|}\bar\lambda\big).$$

Since $\bar\lambda \mapsto K_{t_0}\bar\lambda$ is $\mathbb{R}$-linear, the map

$$\Xi(\phi, y) := \big(\phi \cdot \exp(-2\pi i\, K_{t_0}(B^+_{|t_0|})^{-1}y),\, y\big)$$

conjugates the Γ^0-action to the product action $(\phi, y) \mapsto (\phi, y + B^+_{|t_0|}\bar\lambda)$: indeed, writing $K := K_{t_0}$, $B^+ := B^+_{|t_0|}$,

$$\Xi\big(\bar\lambda \cdot (\phi, y)\big) = \big(\phi\, e^{2\pi i K\bar\lambda}\, e^{-2\pi i K(B^+)^{-1}y}\, e^{-2\pi i K\bar\lambda},\ y + B^+\bar\lambda\big) = \bar\lambda \cdot_{\mathrm{prod}} \Xi(\phi, y).$$

This yields the splitting $F^0_{t_0} \cong T_B \times T_f$ of the Proposition; Ξ alters ϕ by a translation depending on y only, so the above cell description descends verbatim. Thus c_ϑ collapses T_f over each vertex of Θ to a triple point of W, collapses $S^1_{n_k}$ over the interior of e_k onto the double curve $\bar C_k \cong \mathbb{P}^1$, and is injective over $T_B \setminus \Theta$, where it is onto $W \setminus \mathrm{D}$. This is (i).

Two coordinate facts and an interpolation lemma. We first record two coordinate facts, both contained in Lemma 7.7, and an elementary interpolation lemma. Write

$$(Y_{\le\eta})_{>0} := (Y_{\le\eta})_{\ge 0} \setminus (Y_0)_{\ge 0}.$$

(F1) The map $\kappa : q \mapsto (y(q), |t(q)|)$ is a homeomorphism of $(Y_{\le\eta})_{>0}$ onto $\mathbb{R}^2 \times (0, \eta]$, in which $\Psi^+_{\bar\lambda}(y, \rho) = (y + B^+_\rho\bar\lambda, \rho)$ with $B^+_\rho = B_0 + \mathrm{R}_0/\log\rho$, and $S = \kappa^{-1}(\mathbb{R}^2 \times \{\rho_0\})$.

(F2) $\vartheta : \mathbb{R}^2 \to (Y_0)_{\ge 0}$ is a homeomorphism with $\vartheta(\tilde y + B_0\bar\lambda) = \Psi^+_{\bar\lambda}\vartheta(\tilde y)$. Consequently, for a Γ^+-invariant subset $U \subset (Y_{\le\eta})_{\ge 0}$, a continuous map $a : U \to (Y_0)_{\ge 0}$ is Γ^+-equivariant if and only if $\hat a := \vartheta^{-1} \circ a : U \to \mathbb{R}^2$ satisfies

$$\hat a\big(\Psi^+_{\bar\lambda} q\big) = \hat a(q) + B_0\bar\lambda \qquad (\bar\lambda \in \overline{\Lambda}). \tag{7.4}$$

Setting $\ell_\rho(y) := B_0(B^+_\rho)^{-1}y$ (so that $\ell_{\rho_0} = \ell_{t_0}$), one has $\ell_\rho(y + B^+_\rho\bar\lambda) = \ell_\rho(y) + B_0\bar\lambda$; hence $q \mapsto \ell_{|t(q)|}(y(q))$ is a continuous map $(Y_{\le\eta})_{>0} \to \mathbb{R}^2$ satisfying (7.4), the map $\rho \mapsto B^+_\rho$ being continuous on $(0, \eta]$ with values in $GL_2(\mathbb{R})$ (Lemma 7.5).

Interpolation lemma. If $\hat a, \hat b : U \to \mathbb{R}^2$ are continuous and satisfy (7.4), and $w : U \to [0, 1]$ is continuous and Γ^+-invariant, then $\hat h := (1 - w)\hat a + w\hat b$ is continuous and satisfies (7.4); hence $h := \vartheta \circ \hat h : U \to (Y_0)_{\ge 0}$ is continuous and Γ^+-equivariant. Indeed

$$\hat h\big(\Psi^+_{\bar\lambda} q\big) = (1 - w(q))\big(\hat a(q) + B_0\bar\lambda\big) + w(q)\big(\hat b(q) + B_0\bar\lambda\big) = \hat h(q) + B_0\bar\lambda.$$

(In words: ϑ conjugates Γ^+ to a group of translations of $\mathbb{R}^2$, and affine combinations of translation-equivariant maps are translation-equivariant.)

The homotopy. On $U = S$ take $\hat a := \vartheta^{-1} \circ P_1|_S$ — continuous and satisfying (7.4), since P_1 is Γ^+-equivariant with values in $(Y_0)_{\ge 0}$ by (P3), (P4) — and $\hat b := \ell_{\rho_0} \circ y$, and put, for $u \in [0, 1]$,

$$h_u := \vartheta \circ \Big((1 - u)\,\vartheta^{-1}P_1|_S + u\,\ell_{\rho_0} \circ y\Big) : S \longrightarrow (Y_0)_{\ge 0}. \tag{7.5}$$

By the interpolation lemma (constant weight $w \equiv u$) every h_u is Γ^+-equivariant; $(u, q) \mapsto h_u(q)$ is continuous; and

$$h_0 = P_1|_S, \qquad h_1 = \vartheta \circ \ell_{t_0} \circ y.$$

Then $H_u\big((\phi, \arg t_0) \cdot q\big) := (\phi, \arg t_0) \cdot h_u(q)$ ($\phi \in T_f$, $q \in S$) is well defined (points of S have trivial stabiliser), continuous (via the quotient map m), T_f- and Γ^0-equivariant (by (7.2) and the Γ^+-equivariance of h_u), and is a homotopy from $R_1|_{Y_{t_0}}$ to $\tilde c$ through maps $Y_{t_0} \to Y_0$; descending modulo Γ^0, the map $F^0_{t_0} \to W$ induced by $R_1|_{Y_{t_0}}$ is homotopic to c_ϑ (the H_u being Γ^0-equivariant and $Y_0/\Gamma^0 = W$).

The explicit modification: equality for the fixed t_0. We modify the given P itself, on a band of positive torus points around S. Fix δ with $0 < \delta < \rho_0/2$ and let

$$\chi(\rho) := \max\big(0,\ 1 - |\rho - \rho_0|/\delta\big) \qquad (0 < \rho \le \eta),$$

a tent function defined on $(0,\eta]$ (one-sided if $\rho_0=\eta$, which is all that is needed), so that $\chi(\rho_0)=1$ and $\chi(\rho)=0$ for $|\rho-\rho_0|\geq\delta$; explicitly $\chi\equiv 0$ on $(0,\rho_0-\delta]$, so that the function w below extends by 0 continuously to $(Y_0)_{\geq 0}$ (i.e. as $t\to 0$). Put

$$U_\delta := \kappa^{-1}\Big(\mathbb{R}^2\times\big((\rho_0-\delta,\rho_0+\delta)\cap(0,\eta]\big)\Big)\subset (Y_{\leq\eta})_{>0},$$

an open Γ^+-invariant set containing S and disjoint from $(Y_0)_{\geq 0}$ (on it $|t|>\rho_0/2$), and let

$$w(q):=\chi(|t(q)|)\quad (q\in(Y_{\leq\eta})_{>0}),\qquad w:=0\ \text{ on } (Y_0)_{\geq 0}:$$

a continuous, Γ^+-invariant function on $(Y_{\leq\eta})_{\geq 0}$ (the maps Ψ^+_λ preserve t), equal to 1 on S and to 0 outside U_δ. Define $K:(Y_{\leq\eta})_{\geq 0}\times[0,1]\to(Y_0)_{\geq 0}$ by

$$K_u(q):=\begin{cases}\vartheta\Big(\big(1-u\,w(q)\big)\,\vartheta^{-1}\big(P_1(q)\big)+u\,w(q)\,\ell_{|t(q)|}\big(y(q)\big)\Big), & q\in U_\delta,\\ P_1(q), & q\notin U_\delta.\end{cases}\tag{7.6}$$

This is well defined and continuous. Indeed, the first-line formula is defined and continuous on all of the open set $(Y_{\leq\eta})_{>0}$ — there $\vartheta^{-1}P_1$ and $q\mapsto\ell_{|t(q)|}(y(q))$ are continuous $\mathbb{R}^2$-valued maps by (F1), (F2), and w is continuous — and it equals P_1 wherever $w=0$, in particular on $(Y_{\leq\eta})_{>0}\setminus U_\delta$. Together with the open set $\{|t|<\rho_0-\delta\}\supset(Y_0)_{\geq 0}$, on which the second line gives $K_u=P_1$, this is an open cover of $(Y_{\leq\eta})_{\geq 0}$ on whose overlap the two formulas agree; hence K is well defined and continuous. (Equivalently: w vanishes on $\{\big||t|-\rho_0\big|\geq\delta\}\supset\partial U_\delta$, and the closure $\overline{U}_\delta$ lies in $\{|t|\geq\rho_0-\delta\}\subset(Y_{\leq\eta})_{>0}$, so that the first-line formula is defined on $\overline{U}_\delta$; hence K is continuous by pasting along the closed cover $\overline{U}_\delta\cup\big((Y_{\leq\eta})_{\geq 0}\setminus U_\delta\big)$.) Its properties are:

(K1) $K_0=P_1$ (put $u=0$);

(K2) $K_u|_{(Y_0)_{\geq 0}}=P_1|_{(Y_0)_{\geq 0}}=\mathrm{id}$ for all u, since $(Y_0)_{\geq 0}\cap U_\delta=\emptyset$;

(K3) K_u takes values in $(Y_0)_{\geq 0}$, so each K_u is a retraction of $(Y_{\leq\eta})_{\geq 0}$ onto $(Y_0)_{\geq 0}$;

(K4) each K_u is Γ^+-equivariant: outside U_δ it is P_1, and on U_δ this is the interpolation lemma applied with $\hat{a}=\vartheta^{-1}P_1$, $\hat{b}=\ell_{|t|}\circ y$ and the Γ^+-invariant weight $u\,w$;

(K5) $K_1|_S=\vartheta\circ\ell_{\rho_0}\circ y=\vartheta\circ\ell_{t_0}\circ y$, because $w\equiv 1$ on S.

Now put

$$P'_s:=\begin{cases}P_{2s}, & 0\leq s\leq\frac{1}{2},\\ K_{2s-1}, & \frac{1}{2}\leq s\leq 1,\end{cases}$$

continuous since $P_1=K_0$ at $s=\frac{1}{2}$. Then $P'_0=\mathrm{id}$, i.e. (P1); $P'_s|_{(Y_0)_{\geq 0}}=\mathrm{id}$ for all s, i.e. (P2) ((P2) for P when $s\leq\frac{1}{2}$, (K2) when $s\geq\frac{1}{2}$); $P'_1=K_1$ maps $(Y_{\leq\eta})_{\geq 0}$ into $(Y_0)_{\geq 0}$ by (K3), i.e. (P3); every P'_s is Γ^+-equivariant by (P4) and (K4), i.e. (P4); and, by (K5),

$$P'_1|_S=\vartheta\circ\ell_{t_0}\circ y.\tag{7.7}$$

Thus P' again satisfies (P1)–(P4); what has been changed is only the end map P_1, only on the band $U_\delta=\{\,\big||t|-\rho_0\big|<\delta\}$ around S, a set disjoint from $(Y_0)_{\geq 0}$, and only by the straight-line homotopy in the ϑ-chart $\mathbb{R}^2\cong(Y_0)_{\geq 0}$, damped by the bump $\chi(|t|)$.

Applying Lemma 7.9 to P'. Since P' satisfies (P1)–(P4), Lemma 7.9 applies to it verbatim and yields a Γ^0- and T_c-equivariant strong deformation retraction

$$R'_s(\phi\cdot q):=\phi\cdot P'_s(q)\qquad(\phi\in T_c,\ q\in(Y_{\leq\eta})_{\geq 0})$$

of $Y_{\leq\eta}$ onto Y_0, and a strong deformation retraction r'_s of N'_0 onto W induced by $G^{-1}R'_sG$; the descent is continuous because the quotient map $Y_{\leq\eta}\times[0,1]\to N'_0\times[0,1]$ is again a quotient map, being the product of the quotient map $Y_{\leq\eta}\to N'_0$ — which is open, Γ acting properly discontinuously, so that the orbit map is open — with the identity of the locally compact space $[0,1]$. Put $r:=r'_1$.

Equality on Y_{t_0}. Every point of Y_{t_0} is $p = (\phi, \arg t_0) \cdot q$ with $\phi \in T_f$, $q \in S$. Hence, by the definition of R', by (7.7) and by the definition of $\tilde{c}$,

$$R'_1(p) = (\phi, \arg t_0) \cdot P'_1(q) = (\phi, \arg t_0) \cdot \vartheta\big(\ell_{t_0}(y(q))\big) = \tilde{c}(p), \tag{7.8}$$

i.e. $R'_1|_{Y_{t_0}} = \tilde{c}$ exactly.

Equality on F; proof of (ii). Let $\pi : Y_{\leq\eta} \to N'_0 = Y_{\leq\eta}/\Gamma$ and $\pi^0 : Y_{\leq\eta} \to Y_{\leq\eta}/\Gamma^0$ be the quotient maps, so that $F = \pi(Y_{t_0})$ and $F^0_{t_0} = \pi^0(Y_{t_0})$; on $W = \pi(Y_0) = \pi^0(Y_0)$ the two agree, and $G = \mathrm{id}$ there. Since G preserves Y_{t_0} and satisfies $G\Psi_{\bar\lambda} = \Psi^0_{\bar\lambda}G$ (Lemma 7.5(b)), it induces a homeomorphism

$$\bar{G} : F \to F^0_{t_0}, \qquad \bar{G}(\pi p) = \pi^0(Gp),$$

of tori, the identity on the Λ-markings (Lemma 7.5(c)). By definition $r(\pi p) = \pi\big(G^{-1}R'_1G(p)\big)$. For $p \in Y_{t_0}$ we have $G(p) \in Y_{t_0}$, hence $R'_1G(p) = \tilde{c}(Gp) \in Y_0$ by (7.8), and G^{-1} is the identity on Y_0; therefore

$$r(\pi p) = \pi\big(\tilde{c}(Gp)\big) = \pi^0\big(\tilde{c}(Gp)\big) = c_\vartheta\big(\pi^0(Gp)\big) = c_\vartheta\big(\bar{G}(\pi p)\big),$$

using $\pi|_{Y_0} = \pi^0|_{Y_0}$ and the definition of c_ϑ as the map induced by $\tilde{c}$. That is, $r|_F = c_\vartheta \circ \bar{G}$, which is (ii).

Other levels; proof of (iii). The above can be arranged for one level $|t_0|$ at a time, or for any finite set of levels simultaneously by using disjoint bands U_δ; but not for all levels at once, since $\vartheta \circ \ell_{|t|} \circ y$ has no continuous extension by the identity over $(Y_0)_{\geq 0}$ (near the interior of $(E_v)_{\geq 0}$ one has $y \to v$), and the Proposition accordingly fixes t_0. For any other t with $0 < |t| \leq \eta$, the homotopy (7.5), built at the level $|t|$ with P' in place of P (legitimate since P' satisfies (P1)–(P4)), shows that $r|_{f^{-1}(t)}$ is homotopic to $c^{(t)}_\vartheta \circ \bar{G}_t$, where $c^{(t)}_\vartheta : F^0_t \to W$ and $\bar{G}_t : f^{-1}(t) \to F^0_t$ are defined as c_ϑ, $\bar{G}$ with $|t|$ in place of ρ_0; this is what the applications below, all homological, require. □

Proof of Proposition 7.2. Let η_0 be the constant of Lemma 7.5, let $0 < \eta \leq \eta_0$ and fix t_0 with $0 < |t_0| \leq \eta$; put $F = f^{-1}(t_0)$.

By Lemmas 7.7 and 7.8 the quotient Q^+ admits a strong deformation retraction $\bar{P}$ onto H; by Lemma 7.9 its lift P satisfies (P1)–(P4) and produces a Γ^0- and T_c-equivariant strong deformation retraction R of $Y_{\leq\eta}$ onto Y_0, whence, by Lemma 7.5, a $\{\Psi_{\bar\lambda}\}$- and T_f-equivariant strong deformation retraction $G^{-1}R_sG$ of $Y_{\leq\eta}$ onto Y_0 descending to a strong deformation retraction of N'_0 onto W. This is the first assertion of the Proposition (and re-proves Lemma 7.3(i)).

Apply now Lemma 7.10 to this P: part (ii) produces a modified P', again satisfying (P1)–(P4), hence again — by Lemma 7.9 and Lemma 7.5 — a $\{\Psi_{\bar\lambda}\}$- and T_f-equivariant strong deformation retraction of $Y_{\leq\eta}$ onto Y_0 descending to a strong deformation retraction r'_s of N'_0 onto W, and the retraction $r := r'_1$ satisfies $r|_F = c_\vartheta \circ \bar{G}$. By Lemma 7.10(i) (whose ingredients are the stabiliser computation (7.1) of Lemma 7.6 and the honeycomb homeomorphism ϑ of Lemma 7.7) the map c_ϑ, read in the splitting $F^0_{t_0} \cong T_B \times T_f$, collapses T_f over each vertex of Θ, collapses $S^1_{n_k}$ over the interior of e_k onto $\bar{C}_k$ and is injective over $T_B \setminus \Theta$; since $\bar{G}$ is an isomorphism of tori which is the identity on the Λ-markings, the composite $c = r|_F = c_\vartheta \circ \bar{G}$ has exactly the description stated in the Proposition. Finally, the last assertion of the Proposition is Lemma 7.10(iii). □

7.3 The homology of W and the specialisation maps

Proposition 7.11. $H_*(W) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^4, \mathbb{Z}^2, \mathbb{Z})$, torsion free, with $H_2(W) = \langle[\bar{C}_1], [\bar{C}_2], [\bar{C}_3], [\mathbb{T}_B]\rangle$, $\mathbb{T}_B := c(T_B \times \{\mathrm{pt}\})$. Moreover $\pi_1(W) \cong \mathbb{Z}^2$ and $e(W) = 2$.

Proof. Let $\nu : \mathrm{dP}_6 \to W$ be the normalisation (§4); $\mathsf{D} = \bar{C}_1 \cup \bar{C}_2 \cup \bar{C}_3$, three copies of $\mathbb{P}^1$ meeting pairwise exactly at P, Q, so $H_*(\mathsf{D}) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, 0, 0)$ and $e(\mathsf{D}) = 2$.

Let $\partial \subset \mathrm{dP}_6$ be the hexagonal cycle of six (-1)-curves $C_1, \ldots, C_6$. It is *nodal* (six nodes), hence not a submanifold, and we use instead a *regular (plumbing) neighbourhood* $\mathcal{N}$ of ∂ in dP_6: the plumbing of six disc bundles according to the hexagonal cycle, a compact 4-manifold with boundary which deformation retracts onto ∂ and whose interior is an open neighbourhood of ∂. Any such $\mathcal{N}$ will do: since ν is injective off ∂, and the identifications $C_i \to C_{i+3}$ by which ν glues the six curves in opposite pairs take place inside $\partial \subset \mathcal{N}$, the set $\mathcal{N}$ is ν-saturated, and a strong deformation retraction of $\mathcal{N}$ onto ∂ fixing ∂ pointwise is compatible with those identifications; hence both $\mathcal{N}$ and the retraction descend under ν. Put

$$P_0 := \mathrm{dP}_6 \setminus \operatorname{int}\mathcal{N},$$

a compact oriented 4-manifold with boundary. Now ν identifies $\mathrm{dP}_6 \setminus \partial \cong W \setminus \mathsf{D} \cong (\mathbb{C}^*)^2$ and restricts to a homeomorphism of P_0 onto $W \setminus \operatorname{int}\mathcal{N}_W$, where $\mathcal{N}_W := \nu(\mathcal{N})$ is, by the preceding remark, a regular neighbourhood of D in W deformation retracting onto D; moreover $P_0 \hookrightarrow \mathrm{dP}_6 \setminus \partial$ is a deformation retract, so $P_0 \simeq (\mathbb{C}^*)^2 \simeq T^2$.

We compare the pairs homologically, by homotopy equivalence and excision. Since $\mathsf{D} \hookrightarrow \mathcal{N}_W$ is a deformation retract, the five lemma gives $H_k(W,\mathsf{D}) \cong H_k(W,\mathcal{N}_W)$; excising the interior of a smaller regular neighbourhood $\mathcal{N}'_W \subset \operatorname{int}\mathcal{N}_W$ of D gives

$$H_k(W,\mathcal{N}_W) \cong H_k\big(W \setminus \operatorname{int}\mathcal{N}'_W,\ \mathcal{N}_W \setminus \operatorname{int}\mathcal{N}'_W\big) \cong H_k(P_0,\partial P_0),$$

the last isomorphism because the collar between the two neighbourhoods deformation retracts the pair onto $(P_0,\partial P_0)$. Hence

$$H_k(W,\mathsf{D}) \cong H_k(P_0,\partial P_0) \cong H^{4-k}(P_0) = H^{4-k}\big((\mathbb{C}^*)^2\big) = \mathbb{Z},\ \mathbb{Z}^2,\ \mathbb{Z},\ 0$$

for $k = 4,3,2$ and $k \le 1$, the middle isomorphism being Lefschetz duality for the manifold with boundary P_0. Thus (W,D) has one relative 2-cell, two relative 3-cells and one relative 4-cell, dual to the cells of T^2.

Both boundary maps of the pair vanish, because the generators of $H_k(W,\mathsf{D})$ lift to absolute cycles. Indeed, by Proposition 7.2, $\mathbb{T}_B = c(T_B \times \{\mathrm{pt}\})$ meets $W \setminus \mathsf{D}$ in $(T_B \setminus \Theta) \times \{\mathrm{pt}\}$, which represents the generator of $H_2(W,\mathsf{D}) \cong H^2(P_0)$: under the Lefschetz duality $H_2(P_0,\partial P_0) \cong H^2(P_0)$ the pairing with $H_2(P_0) \cong H_2(T^2) = \mathbb{Z}$ is the intersection pairing, the generator of $H_2(P_0)$ is the compact torus $T_f \subset W \setminus \mathsf{D} \cong (\mathbb{C}^*)^2$, and in $(\mathbb{C}^*)^2 \cong (T_B \setminus \Theta) \times T_f$ the "Log-section" $(T_B \setminus \Theta) \times \{\mathrm{pt}\}$ meets the level torus $\{y = y_0\}$ transversally in one point, so the intersection number is ± 1; as the pairing $H_2(P_0,\partial P_0) \times H_2(P_0) \to \mathbb{Z}$ is perfect between free groups of rank 1, a class pairing to ± 1 is a generator. Likewise the two 3-cycles $c(T_B \times S^1_{\hat{w}})$, $c(T_B \times S^1_{\hat{\delta}})$ meet $W \setminus \mathsf{D}$ in the relative cycles $(T_B \setminus \Theta) \times S^1_\nu$ ($\nu = \hat{w}, \hat{\delta}$), whose intersection numbers with the circles $\{y_0\} \times S^1_\mu$ generating $H_1(P_0) = H_1(T_f)$ are $\pm\det(\nu,\mu)$, a unimodular matrix; so they represent a basis of $H_3(W,\mathsf{D}) \cong H^1(P_0) = H^1(T^2)$. Hence

$$H_1(W) \cong H_1(\mathsf{D}) = \mathbb{Z}^2, \qquad 0 \to H_2(\mathsf{D}) \to H_2(W) \to \mathbb{Z} \to 0,$$
$$H_3(W) \cong \mathbb{Z}^2, \qquad H_4(W) = \mathbb{Z},$$

all extensions being of free groups, which gives the stated groups and generators.

Finally $\pi_1(W) \cong \pi_1(N'_0) = \overline{\Lambda}$. By Lemma 7.3(i) the inclusion $W \hookrightarrow N'_0$ is a deformation retract, and the computation of π_1 in Lemma 7.4 is stated for N_0, so we argue with N'_0 directly: $N'_0 = Y_{\le\eta}/\Gamma$ is a free properly discontinuous quotient, and its covering space $Y_{\le\eta}$ is simply connected, as follows. By Lemma 7.9 the inclusion $Y_0 \hookrightarrow Y_{\le\eta'}$ is that of a strong deformation retract for *every* $0 < \eta' \le \eta_0$. Shrinking ε so that $\varepsilon \le \eta_0$ (this leaves N'_0, W and the statement unchanged, and Corollary 4.8 holds for every ε), we may write $Y_\varepsilon = \bigcup_{\eta'<\varepsilon} Y_{\le\eta'}$ as an increasing union of closed subspaces, each containing Y_0 and retracting onto it; consequently each inclusion $Y_{\le\eta'} \hookrightarrow Y_{\le\eta''}$ ($\eta' \le \eta''$) induces an isomorphism on π_1, both composites with $Y_0 \hookrightarrow (\cdot)$ being isomorphisms. Since π_1 commutes with such increasing unions, the inclusions induce isomorphisms

$$\pi_1(Y_{\le\eta}) \cong \pi_1(Y_0) \cong \pi_1(Y_\varepsilon) = 1$$

by Corollary 4.8. (Equivalently, $N'_0 \hookrightarrow N_0$ is a homotopy equivalence and one may quote Lemma 7.4.) And $e(W) = e(\mathrm{dP}_6) - e(\partial) + e(\mathsf{D}) = 6 - 6 + 2 = 2$. □

Proposition 7.12. For every q the map $c_* : H_q(F) \to H_q(W)$ is surjective with kernel $(\bigwedge^q M_0 - I)\bigwedge^q \Lambda$; dually

$$\mathrm{sp}^q := c^* : H^q(W) \longrightarrow H^q(F) = \textstyle\bigwedge^q V$$

is injective with image $(\bigwedge^q V)^{T_0}$. Explicitly

$$V^{T_0} = \langle \gamma, u \rangle, \qquad (\textstyle\bigwedge^3 V)^{T_0} = \langle \gamma u w,\ \gamma u \delta \rangle,$$
$$(\textstyle\bigwedge^2 V)^{T_0} = \langle \gamma u,\ \gamma\delta,\ uw,\ \gamma w - u\delta \rangle,$$

and $\mathrm{sp}^2(\mathbb{T}_B^\vee) = \gamma \wedge u$, $\mathrm{sp}^2(\bar{C}_k^\vee) = \pm\big(b_k^2\,\gamma\delta - a_k^2\,uw + a_k b_k(\gamma w - u\delta)\big)$, where $\nu_k := \det(n_k,\cdot) = (a_k, b_k)$ in the basis (w,δ) of $\Lambda_{\rm tor}^\vee$. In degree 2 the two factors of the splitting $F^0_{t_0}\cong T_B\times T_f$ of Proposition 7.2 have $H_1(T_f)=\Lambda_{\rm tor}=\langle\hat w,\hat\delta\rangle$ and $H_1(T_B\times\{pt\})=\langle\hat\gamma,\hat u\rangle\subset\Lambda$ exactly (not merely modulo $\Lambda_{\rm tor}$). Indeed, in the coordinates (ϕ,y) of $Y_{t_0}\cong T_f\times\mathbb{R}^2$ the section $T_B\times\{\phi_0\}$ is the graph $\phi=\phi_0\exp(2\pi iK_{t_0}(B^+_{|t_0|})^{-1}y)$ (Lemma 7.10, the map Ξ), and along the loop $y\mapsto y_0+sB^+_{|t_0|}\bar\lambda$, $s\in[0,1]$, which represents $\bar\lambda\in\overline{\Lambda}=H_1(T_B)$, the phase advances by exactly $K_{t_0}\bar\lambda=\mathrm{Re}(Z^0_{t_0}\bar\lambda)$; so the lift of this loop to the universal cover of $F^0_{t_0}$ has displacement $Z^0_{t_0}\bar\lambda$ with no $\Lambda_{\rm tor}$-component (compare the formula for the Γ^0-action, $\bar\lambda\cdot(\phi,y)=(\phi e^{2\pi iK_{t_0}\bar\lambda},\,y+B^+_{|t_0|}\bar\lambda))$, and in the marking $\Pi^0_{t_0}=[Z^0_{t_0}\mid I]$ its class is the basis vector $\bar\lambda\in\langle\hat\gamma,\hat u\rangle$. We orient the factors so that $[T_B\times\{pt\}]=\hat\gamma\wedge\hat u$ and $[T_f]=\hat w\wedge\hat\delta$ in $\wedge^2\Lambda=H_2(F)$; these classes are transported to F by $\bar G^{-1}$, which is the identity on $H_1=\Lambda$ (Lemma 7.5(c)). With this choice of orientation the image in $H_2(W,\mathsf{D})\cong\mathbb{Z}$ of the class $[\mathbb{T}_B]=c_*[T_B\times\{\mathrm{pt}\}]\in H_2(W)$ is a generator (Proposition 7.11), and with respect to it the mixed classes $\beta\otimes v$ ($\beta\in H_1(T_B)$, $v\in H_1(T_f)$) have vanishing $\mathbb{T}_B$-component in $H_2(W)$: they are carried into D.

Proof. Since $T_0=I+N$ with $N\gamma=Nu=0$, $Nw=-u$, $N\delta=\gamma$, direct computation on $\bigwedge^2 V$ gives

$$\gamma u\mapsto\gamma u,\quad \gamma w\mapsto\gamma w-\gamma u,\quad \gamma\delta\mapsto\gamma\delta,$$
$$uw\mapsto uw,\quad u\delta\mapsto u\delta-\gamma u,\quad w\delta\mapsto w\delta-\gamma w-u\delta+\gamma u,$$

whence the stated invariants and $(\bigwedge^2 T_0-I)\bigwedge^2 V=\langle\gamma u,\ \gamma w+u\delta\rangle$; degrees $1,3$ are similar. The invariant lattices have ranks $2,4,2$ in degrees $1,2,3$, i.e. exactly $b_q(W)$.

Now c_* in the model of Proposition 7.2. In degree 1 every circle of T_f can be pushed into a vertex of Θ, where it collapses, while $T_B\xrightarrow{\sim}\mathbb{T}_B$; thus $c_*:\Lambda\to H_1(W)=\overline{\Lambda}$ is the projection.

In degree 2, the splitting identifies $H_2(F)=\bigwedge^2\Lambda$ with $\bigwedge^2\overline{\Lambda}\oplus(\overline{\Lambda}\otimes\Lambda_{\rm tor})\oplus\bigwedge^2\Lambda_{\rm tor}$, the outer summands being generated by $[T_B\times\{\mathrm{pt}\}]=\hat\gamma\wedge\hat u$ and $[T_f]=\hat w\wedge\hat\delta$ as stated. One has $c_*[T_f]=0$ (the fibre torus collapses to a point over a vertex of Θ) and $c_*[T_B\times\{\mathrm{pt}\}]=[\mathbb{T}_B]$ by definition. For $\beta\in H_1(T_B)=\overline{\Lambda}$ and $v\in H_1(T_f)=\Lambda_{\rm tor}$ we claim

$$c_*(\beta\otimes v)=\sum_{k=1}^{3}\nu_k(B_0\beta)\,\nu_k(v)\,[\bar C_k],\tag{7.9}$$

$B_0:\overline{\Lambda}\xrightarrow{\sim}\Lambda_{\rm tor}$ being the unimodular shear of §2; in particular the mixed classes have no $[\mathbb{T}_B]$-component, as asserted. This is read off the cell description of Proposition 7.2, using a cellular representative of β (the edges e_k are not cycles, so a cellular representative, rather than a count of transversal crossings of Θ, is what computes the class). Give T_B the CW structure with 1-skeleton Θ (two vertices, the three edges e_k, each oriented from the first vertex to the second) and a single 2-cell, namely the open hexagon $T_B\setminus\Theta$ dual to the unique vertex of $\mathcal{T}_{A_2}/\Lambda_{\rm tor}$; its attaching word traverses each e_k twice with opposite signs, so the cellular ∂_2 vanishes and

$$H_1(T_B)=H_1(\Theta)=\Big\{\textstyle\sum_k m_ke_k:\ \sum_k m_k=0\Big\}.$$

Thus β is represented by a unique cellular cycle $\sum_k m_k(\beta)e_k$. To identify $m_k(\beta)$, let $f_k\subset T_B$ be the image of a triangulation edge of direction n_k; it is a *closed* loop (all vertices of $\mathcal{T}_{A_2}$ are identified in T_B), of homology class $n_k\in\Lambda_{\rm tor}=H_1(T_B)$, and by duality it meets Θ transversally in exactly one point, an interior point of e_k. Intersecting the two cycles β and f_k in T_B therefore gives

$$m_k(\beta)=\pm\,\beta\cdot f_k=\pm\det(B_0\beta,n_k)=\pm\nu_k(B_0\beta),$$

the class of β in $H_1(T_B)=\Lambda_{\rm tor}$ being $B_0\beta$ and the intersection form being det in a basis of $\Lambda_{\rm tor}$; orienting the edges e_k suitably (the coefficient functional m_k, like $\nu_k=\det(n_k,\cdot)$, is defined only up to the sign fixed by an orientation of e_k, resp. of n_k) we may and do take $m_k(\beta)=\nu_k(B_0\beta)$. Consequently $\beta\otimes v$ is represented by the 2-cycle $\sum_k\nu_k(B_0\beta)\,(e_k\times S^1_v)$ in $F=T_B\times T_f$, a cycle supported over Θ and hence carried by c into D (this is the vanishing of its $[\mathbb{T}_B]$-component). By Proposition 7.2, c maps $e_k\times T_f$ onto $\bar C_k$, collapsing $S^1_{n_k}$ over e_k° and all of T_f over the two endpoints; so the annulus $e_k\times S^1_v$ is carried onto $\bar C_k\cong S^2$ by a map which collapses both boundary circles to the triple points P,Q and equals, over e_k°, the product of $\mathrm{id}_{e_k^\circ}$ with $S^1_v\hookrightarrow T_f\to T_f/S^1_{n_k}$. Since $H_1(T_f/S^1_{n_k})=\Lambda_{\rm tor}/\mathbb{Z}n_k\cong\mathbb{Z}$ via $\det(n_k,\cdot)=\nu_k$, that map has degree

$\pm\nu_k(v)$ with respect to the complex orientation of $\bar{C}_k$, with a sign depending only on k, which we absorb into the orientation of $[\bar{C}_k]$; thus $c_*[e_k \times S^1_v] = \nu_k(v)[\bar{C}_k]$, and (7.9) follows.

Surjectivity in degree 2: as B_0 is unimodular it suffices that $\Lambda_{\mathrm{tor}} \otimes \Lambda_{\mathrm{tor}} \to \mathbb{Z}^3$, $x \otimes v \mapsto (\nu_k(x)\nu_k(v))_k$, be onto: with $n_1 = \hat{w}$, $n_2 = \hat{\delta} - \hat{w}$, $n_3 = -\hat{\delta}$ we get $\nu_1 = (0,1)$, $\nu_2 = (-1,-1)$, $\nu_3 = (1,0)$, and $(x,v) = (\hat{w},\hat{w}), (\hat{\delta},\hat{\delta}), (\hat{w},\hat{\delta})$ give $(0,1,1), (1,1,0), (0,1,0)$, of determinant ± 1; together with $c_*[T_B \times \{\mathrm{pt}\}] = [\mathbb{T}_B]$ this gives all of $H_2(W)$. In degree 3, $c_*[T_B \times S^1_v] = v \in \Lambda_{\mathrm{tor}} \cong H_3(W)$ and classes containing $[T_f]$ die, so c_* is onto; degrees $0, 4$ are clear.

Finally $c_* \circ \bigwedge^q M_0 = c_*$: the geometric monodromy h of f over Δ^*_ε becomes homotopic to the identity *after inclusion into* N'_0. Indeed, the fibre transport of F once around the circle $|t| = |t_0|$ inside N'_0 is a homotopy $F \times [0,1] \to N'_0$ from the inclusion $F \hookrightarrow N'_0$ to its composite with h (it is a homotopy inside $N'_0 \simeq W$, not an isotopy of F, and no flow on all of N'_0 is claimed); composing with the retraction r of N'_0 onto W gives $c \simeq c \circ h$. Hence $(\bigwedge^q M_0 - I) \bigwedge^q \Lambda \subseteq \ker c_*$; both are saturated (the first by the computation above, the second because $H_q(W)$ is free) of rank $\binom{4}{q} - b_q(W)$, hence equal. Dualising (all groups free) gives the statements for sp^q, the displayed formulas being the transposes. $\square$

A second proof of the injectivity of sp^q and of $\operatorname{im} \mathrm{sp}^q = (\wedge^q V)^{T_0}$, independent of the retraction of §7.2, by nearby cycles on W, is given in Appendix B (Theorem B.1); the identification $\mathrm{sp}^q = c^*$, the generators of $H_2(W)$ and the degree-2 normalisations above are not re-proved there.

7.4 The multiple fibres, and the index 2 at p_2

Recall $\Lambda^{A_1} = \langle \varepsilon, \hat{\delta} \rangle$, $\varepsilon = \hat{\gamma} + 2\hat{u} - 4\hat{w}$, $\Lambda^{A_2} = \langle \varepsilon', \hat{\delta} \rangle$, $\varepsilon' = \hat{\gamma} + 3\hat{u} - 3\hat{w}$. Call $(v_1, v_2) \in \Lambda^{A_1} \times \Lambda^{A_2}$ *admissible* if $3 \nmid \ell_1$ and ℓ_2 is odd, $\ell_j := \gamma(v_j)$; by §5 this is exactly freeness of $\langle \tilde{g}_j^{\log} \rangle$. Our data $(v_1, v_2) = (\varepsilon, -\varepsilon')$ give $(\ell_1, \ell_2) = (1, -1)$. Note that admissibility says exactly $\gcd(\ell_j, m_j) = 1$.

Lemma 7.13. Let (v_1, v_2) be admissible. Then

$$(A_1 - I)\Lambda = \{a\hat{\gamma} + b\hat{u} + c\hat{w} + d\hat{\delta} :\ a = 0,\ 2b + c + 3d = 0\},$$
$$(A_2 - I)\Lambda = \{a\hat{\gamma} + b\hat{u} + c\hat{w} + d\hat{\delta} :\ a = 0,\ b + c + 2d = 0\},$$

both saturated of rank 2, and $\Lambda_{A_j} := \Lambda/(A_j - I)\Lambda \cong \mathbb{Z}^2$ via $\lambda \mapsto (\gamma(\lambda), \psi_j(\lambda))$ with $\psi_1 = 2u + w + 3\delta$, $\psi_2 = u + w + 2\delta$. Consequently

$$H_1(S_j) = \big(\Lambda_{A_j} \oplus \mathbb{Z}\hat{g}_j\big) / \big(m_j \hat{g}_j - [v_j]\big) \cong \mathbb{Z}^2$$

is torsion free, $H_*(S_j) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z}^2, \mathbb{Z})$, and $H^2(S_j) \cong \mathbb{Z}^2$ carries a unimodular even form of determinant -1.

Proof. The descriptions of $(A_j - I)\Lambda$ follow by direct computation from $A_1\hat{\gamma} = \hat{\gamma} + 6\hat{u} - 6\hat{w} - 2\hat{\delta}$, $A_1\hat{u} = -\hat{w} + \hat{\delta}$, $A_1\hat{w} = \hat{u} - \hat{w}$, $A_1\hat{\delta} = \hat{\delta}$ and the analogous formulas for A_2 (§2); the annihilator of $(A_j - I)\Lambda$ is $\langle \gamma, \psi_j \rangle$ and $(\gamma, \psi_j) : \Lambda \to \mathbb{Z}^2$ is onto. The presentation of $H_1(S_j)$ is the abelianisation of $\pi_1(S_j)$ (Lemma 7.4); it is torsion free iff $\gcd(\ell_j, \psi_j(v_j), m_j) = 1$, which holds since $3 \nmid \ell_1$, ℓ_2 odd. As S_j is bielliptic (§5), $b_1 = b_2 = 2$, $e = 0$ [BHPV, Ser]; the torsion of $H_2(S_j)$ equals that of $H_1(S_j) = 0$, so $H^2(S_j) \cong \mathbb{Z}^2$ is unimodular, and even, since K_{S_j} is numerically trivial and $x^2 \equiv x \cdot K_{S_j} \pmod 2$ by Wu's formula; an even unimodular lattice of rank 2 is the hyperbolic plane, hence of signature 0 and determinant -1. By universal coefficients $H^q(S_j)$ is torsion free for all q. $\square$

Proposition 7.14. Let $\pi_j : F \to S_j$ be the m_j-fold covering. Then $\pi_j^* : H^q(S_j) \to H^q(F)$ is injective with image of finite index $k_j^{(q)}$ in $(\bigwedge^q V)^{T_j}$, where

$$k_1^{(1)} = 3, \qquad k_1^{(2)} = 1, \qquad k_1^{(3)} = 1, \qquad k_1^{(4)} = 3,$$
$$k_2^{(1)} = 4, \qquad k_2^{(2)} = 2, \qquad k_2^{(3)} = 2, \qquad k_2^{(4)} = 4.$$

The cokernels in degrees $1, 2$ are cyclic, generated by the classes of γ and of $q := uw + 6\gamma\delta$; the cokernel in degree 3 is cyclic generated by the class of γuw whenever $4 \mid \varepsilon_2(v_2)$, in particular for our data $v_2 = -\varepsilon'$.

Explicitly, with $\varepsilon_1 = 2u + w + 3\delta$, $\varepsilon_2 = u + w + 2\delta$,

$$\pi_1^* H^1(S_1) = \langle 3\gamma, \varepsilon_1 \rangle,$$
$$\pi_2^* H^1(S_2) = \{x\gamma + y\varepsilon_2 : \ 4 \mid x\ell_2 + y\,\varepsilon_2(v_2)\}.$$

Proof. Injectivity holds as $\pi_{j*}\pi_j^* = m_j$ and $H^q(S_j)$ is torsion free (Lemma 7.13); the image lies in the invariants by equivariance.

Degree 1. A homomorphism $\pi_1(S_j) \to \mathbb{Z}$ is a pair (ξ, n), $\xi \in V^{T_j}$, $m_j n = \xi(v_j)$, pulling back to ξ; so $\pi_j^* H^1(S_j) = \{\xi \in V^{T_j} : m_j \mid \xi(v_j)\}$, with $V^{T_1} = \langle \gamma, \varepsilon_1 \rangle$, $V^{T_2} = \langle \gamma, \varepsilon_2 \rangle$ (§2). As $\varepsilon_1(\varepsilon) = 0$, $\varepsilon_1(\hat{\delta}) = 3$, for $v_1 = a\varepsilon + b\hat{\delta}$ one has $(x\gamma + y\varepsilon_1)(v_1) = x\ell_1 + 3yb$, and $3 \mid \xi(v_1) \iff 3 \mid x$ since $3 \nmid \ell_1$: index 3, cokernel $[\gamma]$. Similarly $\varepsilon_2(\varepsilon') = 0$, $\varepsilon_2(\hat{\delta}) = 2$ give the stated sublattice, of index 4 with cokernel $[\gamma]$ since ℓ_2 is a unit mod 4.

Degree 2. With values in $\bigwedge^4 V = \mathbb{Z}\,\mathrm{vol}$, $\mathrm{vol} = \gamma u w \delta$, one has $(\bigwedge^2 V)^{T_1} = \langle \gamma\varepsilon_1, q \rangle$, $(\bigwedge^2 V)^{T_2} = \langle \gamma\varepsilon_2, q \rangle$ with $\gamma\varepsilon_1 \wedge q = 3\,\mathrm{vol}$, $\gamma\varepsilon_2 \wedge q = 2\,\mathrm{vol}$, $q \wedge q = 12\,\mathrm{vol}$: Gram matrices $\left(\begin{smallmatrix} 0 & 3 \\ 3 & 12 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 12 \end{smallmatrix}\right)$, determinants $-9, -4$. Since $\langle \pi_j^* a, \pi_j^* b \rangle = m_j \langle a, b \rangle$ and $H^2(S_j)$ is unimodular, $(k_j^{(2)})^2 = m_j^2/|\det|$, i.e. $9/9 = 1$ and $16/4 = 4$. If $q = \pi_2^* a$ then $12 = q \wedge q = 4(a \cdot a)$, so $a \cdot a = 3$, contradicting evenness (Lemma 7.13); hence the cokernel $\mathbb{Z}/2$ is generated by $[q]$.

Degrees 3, 4. In degree 4, π_j^* is multiplication by m_j. In degree 3 use the perfect pairing $H^1 \times H^3 \to \mathbb{Z}$: $(\bigwedge^3 V)^{T_1} = \langle \gamma u w, \zeta_1 \rangle$, $\zeta_1 = -uw\delta + 2\gamma w\delta + 4\gamma u\delta$; $(\bigwedge^3 V)^{T_2} = \langle \gamma u w, \zeta_2 \rangle$, $\zeta_2 = -uw\delta + 3\gamma w\delta + 3\gamma u\delta$; the pairings with (γ, ε_j) have matrices $\left(\begin{smallmatrix} 0 & -1 \\ -3 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ -2 & 0 \end{smallmatrix}\right)$, determinants $-3, -2$, so $k_j^{(1)} k_j^{(3)} |\det| = m_j^2$ gives $k_1^{(3)} = 1$, $k_2^{(3)} = 2$. For the generator of the degree-3 cokernel at p_2 assume $4 \mid \varepsilon_2(v_2)$; then $\varepsilon_2 \in \pi_2^* H^1(S_2)$ by the degree-1 description (for $v_2 = -\varepsilon'$ one has $\varepsilon_2(v_2) = 0$, so this holds). All pairings of $\pi_2^* H^1(S_2)$ with $\pi_2^* H^3(S_2)$ lie in $4\mathbb{Z}$, being $m_2 = 4$ times pairings on S_2, while $\langle \varepsilon_2, \gamma u w \rangle = -2 \notin 4\mathbb{Z}$; hence $\gamma u w \notin \pi_2^* H^3(S_2)$ and the cokernel $\mathbb{Z}/2$ is generated by $[\gamma u w]$. □

Remark 7.15. For a map φ of free finitely generated abelian groups, $\mathrm{coker}(\varphi^\vee) \cong \mathrm{Ext}(\mathrm{coker}\,\varphi, \mathbb{Z}) \oplus \mathrm{Hom}(\ker \varphi, \mathbb{Z})$; so $\pi_{2*} : H_2(F) \to H_2(S_2)$ has image of index 2, whereas π_{1*} is onto. This produces $H^0(B, R^2 f_* \mathbb{Z}) = \mathbb{Z} \cdot 2q$ rather than $\mathbb{Z} \cdot q$ in Proposition 7.26.

7.5 The fundamental group

By §6, $X = J \cup N_0 \cup N_1 \cup N_2$. Write $\pi_1(B^\circ) = F(a_1, a_2)$, $a_0 := (a_1 a_2)^{-1}$ (the a_i being the clockwise meridians of Convention 3.17), and let $\hat{\rho}_i \in \pi_1(J)$ be the lift of a_i along the holomorphic zero-section, so

$$\pi_1(J) = \Lambda \rtimes F(\hat{\rho}_1, \hat{\rho}_2), \qquad \hat{\rho}_i \lambda \hat{\rho}_i^{-1} = A_i \lambda, \qquad \hat{\rho}_1 \hat{\rho}_2 \hat{\rho}_0 = 1.$$

Two identifications enter below, one at p_1, p_2 and one at p_0. Both are computed from the gluings of §6, and a different answer would change the homotopy type: altering a gluing by a fibrewise translation t_λ ($\lambda \in \Lambda$) changes ℓ_j by $m_j \gamma(\lambda)$ (with $m_0 := 1$), hence changes $p = 12\ell_0 - 4\ell_1 - 3\ell_2$ of Theorem 7.17 by $\pm 12\gamma(\lambda)$. The computations are the following.

At p_j. Let $U = A_0^{(j)} \times \Delta_{s_j} \to N_j$ be the covering of Lemma 7.4, with deck generator

$$\tilde{g}_j^{\log}(x, z) = (A_j x + v_j/m_j,\ \zeta z), \qquad \zeta = e^{-2\pi i/m_j}$$

(the disc factor as forced by $s_j \circ g_j = \zeta_j s_j$, §5), and $\hat{g}_j \in \pi_1(M_j)$ the lift of $\tilde{g}_j^{\log}$ fixed there, so that $\hat{g}_j^{m_j} = t_{v_j}$ in $\pi_1(S_j)$ (Proposition 5.6: $(\tilde{g}_j^{\log})^{m_j} = t_{v_j}$). By Lemma 6.5(ii) the holomorphic zero-section ς_j is $\tilde{g}_j^{\log}$-invariant; consequently the transport of ς_j once around the meridian of p_j is a *generator of the deck group*, with no residual fibre translation, so that the gluing G_j carries $\hat{\rho}_j$ to $\hat{g}_j^{\varepsilon}$ with $\varepsilon \in \{\pm 1\}$. The relation is

$$\hat{\rho}_j^{m_j} = t_{\varepsilon v_j} \qquad (j = 1, 2)$$

with $\varepsilon = \pm 1$ a global sign fixed by Convention 3.17; no further fibre translation enters, by Lemma 6.5(ii). Here ε records only *which* generator, and is determined by the orientation of the meridians: the meridians used here and in §3 are the *clockwise* ones. With those meridians one has $\varepsilon_1 = \varepsilon_2 = +1$; and — this is the point — the equality $\varepsilon_1 = \varepsilon_2$ is not a convention at all, but is forced simultaneously at p_1 and at p_2 by Lemma 7.16 below. Consequently the relative sign of ℓ_1 and ℓ_2 in p is fixed once and for all, and $|p| = |12\ell_0 - 4\ell_1 - 3\ell_2|$ is independent of every orientation convention. The same convention orients the toric meridian at p_0, so ε enters ℓ_0, ℓ_1, ℓ_2 by one and the same global sign: passing from the clockwise to the counterclockwise meridians replaces (ℓ_0, ℓ_1, ℓ_2) by $(-\ell_0, -\ell_1, -\ell_2)$ and p by $-p$. This changes neither $|p|$ nor any statement below, Theorems 7.17 and 7.22 depending on p only through $|p|$. With Convention 3.17, then,

$$\hat{\rho}_j^{m_j} = t_{v_j} \qquad (j = 1, 2),$$

consistently with the presentation of Lemma 7.4. The content of the preceding paragraph is that no further fibre translation t_λ intervenes, and that the *relative* signs of ℓ_0, ℓ_1, ℓ_2 are those displayed; this is what fixes ℓ_1, ℓ_2, hence $|p|$ below.

At p_0. The toric meridian differs from $\hat{\rho}_0$ by a fibre translation, $\tilde{a}_0 = t_\mu \hat{\rho}_0$ with $\mu \in \Lambda/\Lambda_{\mathrm{tor}}$ unique; set $\ell_0 := \gamma(\mu)$ (well defined, as γ vanishes on Λ_{tor}). For the tautological gluing G_0 of §6 one has $\mu = 0$, i.e. $\ell_0 = 0$. Indeed, by Proposition 4.7(iv) together with Lemma 6.5(i), the holomorphic zero-section over the punctured disc at p_0 corresponds, under the toric description of §4, to the closed disc through the point $(1,1)$, i.e. to $\{x = (1,1),\ |t| \le \eta\} \subset Y_\varepsilon$; its boundary circle over $|t| = \eta$ is precisely the toric meridian $\tilde{a}_0$ of Lemma 7.4 (oriented clockwise, by the same Convention 3.17), whence $\hat{\rho}_0 = \tilde{a}_0$ and $\mu = 0$. A general fibrewise affine regluing at p_0 shifts ℓ_0 as described above and so realises every $\ell_0 \in \mathbb{Z}$.

Thus, inside the common group $\pi_1(J) = \Lambda \rtimes F(\hat{\rho}_1, \hat{\rho}_2)$, the three meridians of p_1, p_2, p_0 are

$$\mathfrak{m}_1 = \hat{\rho}_1, \qquad \mathfrak{m}_2 = \hat{\rho}_2, \qquad \mathfrak{m}_0 = \tilde{a}_0 = t_\mu \hat{\rho}_0 = t_\mu \hat{\rho}_2^{-1} \hat{\rho}_1^{-1}, \tag{7.10}$$

the gluings G_1, G_2, G_0 identifying them with $\hat{g}_1 \in \pi_1(M_1)$, $\hat{g}_2 \in \pi_1(M_2)$ and the toric meridian of $\pi_1(M)$ respectively.

Lemma 7.16 (Sign lemma)**.** For $j = 1, 2$ let $\varepsilon_j \in \{\pm 1\}$ be defined by the relation that the filling N_j imposes on the zero-section lift $\hat{\rho}_j$ of the *clockwise* meridian of p_j,

$$\hat{\rho}_j^{m_j} = t_{\varepsilon_j v_j} \qquad \text{in } \pi_1(N_j \setminus m_j S_j) \to \pi_1(N_j), \qquad m_1 = 3,\ m_2 = 4.$$

Then $\varepsilon_1 = \varepsilon_2$, and with the conventions of this paper $\varepsilon_1 = \varepsilon_2 = +1$. Consequently, for the tautological gluing at p_0 ($\ell_0 = 0$),

$$|\pi_1(X)| = |H_1(X)| = \big|\, 4\gamma(v_1) + 3\gamma(v_2) \,\big|$$

independently of every orientation convention: for our data $(v_1, v_2) = (\varepsilon, -\varepsilon')$ this is $|4 - 3| = 1$, and for the comparison threefold X' with $(v_1, v_2) = (\varepsilon, +\varepsilon')$ it is $|4 + 3| = 7$.

Proof. A sign could enter at four places: (S1) the rotation sense of g_j at z_j; (S2) the translation part $+v_j/m_j$ of $\tilde{g}_j^{\log}$; (S3) the regluing section ς_j and its drift; (S4) the identification of the meridian lift with a power of the deck generator. We show that at each place the *same* sign enters for $j = 1$ and for $j = 2$; only (S1) has mathematical content, (S2)–(S4) being the observation that §5–§7.5 apply one formula with j as a parameter.

(S1) The two elliptic generators rotate in the same sense, and this is forced. By Proposition 2.11(ii), Δ is the orientation-preserving subgroup of the reflection group of the hyperbolic triangle ABC with angles $\pi/3, \pi/4, 0$ and vertices in counterclockwise order; $x = \sigma_b \sigma_c$ and $y = \sigma_c \sigma_a$ are the *counterclockwise* rotations about A and B by $2\pi/3$ and $2\pi/4$, $z = \sigma_a \sigma_b$ is parabolic, $xyz = 1$, and $\Delta = \langle x \rangle * \langle y \rangle \cong \mathbb{Z}/3 * \mathbb{Z}/4$. The paper sets $g_1 := x^{-1}$, $g_2 := y^{-1}$ — the two rotations are reversed simultaneously — so that $g_1'(z_1) = e^{-2\pi i/3} = \zeta_1$, $g_2'(z_2) = e^{-2\pi i/4} = \zeta_2$ and $g_0 := (g_1 g_2)^{-1}$ is parabolic (conjugate to z^{-1}).

That "simultaneously" is forced. Since the quotient has exactly one cusp (Proposition 2.11(iii); cf. [Bea83, Ch. 10]), the parabolic elements of Δ are exactly the conjugates of the non-zero powers of $z = (xy)^{-1}$, i.e. of the cyclically reduced words $(xy)^n$ and $(y^{-1}x^{-1})^n = (y^3x^2)^n$ $(n \ge 1)$; and two cyclically reduced words in a free product are conjugate if and only if they are cyclic permutations of one another. Suppose one tried the mixed assignment $(g_1, g_2) = (x^{-1}, y)$. Then $g_0 = (g_1 g_2)^{-1} = y^{-1}x = y^3 x$, a cyclically reduced word of syllable length 2, which is not a cyclic permutation of xy (permutations xy, yx) nor of y^3x^2 (permutations y^3x^2, x^2y^3), hence not conjugate to any $(xy)^{\pm n}$: g_0 is not parabolic. The symmetric mixed assignment $(g_1, g_2) = (x, y^{-1})$ is excluded in the same way: there $g_0 = (xy^{-1})^{-1} = yx^{-1} = yx^2$, again cyclically reduced of syllable length 2 and not a cyclic permutation of xy or of y^3x^2, hence not conjugate to any $(xy)^{\pm n}$. (Conjugates $hy^{\pm 1}h^{-1}$ are not at issue here, since g_1 and g_2 are the rotations about the vertices A and B of the *fixed* fundamental triangle, not arbitrary conjugates of them.) Geometrically, $\mathrm{rot}(A, -2\pi/3) \circ \mathrm{rot}(B, +2\pi/4)$ is hyperbolic. Since §3 needs g_0 parabolic — $\tau \circ g_0 = \tau - 1$, and g_0 generates the cusp stabiliser used to build the coordinate s at p_0 (Proposition 2.11(iii)) — the rotation senses of g_1 and g_2 must agree.

Which sense is then fixed at z_1. The law $\tau \circ g_1 = R_1 \circ \tau$ with $R_1(\tau) = 1 - 1/\tau$, τ a local biholomorphism at z_1 with $\tau(z_1) = \rho = e^{i\pi/3}$ (Proposition 3.6: $\tau - \rho$ has order 1 at z_1), gives on differentiating $g_1'(z_1) = R_1'(\rho) = \rho^{-2} = e^{-2\pi i/3}$: clockwise. At z_2 no independent check is available from τ alone: there $\tau - i$ vanishes to order 2, so from $\tau \circ g_2 = S \circ \tau$ with $S(\tau) = -1/\tau$ one obtains, comparing the quadratic terms, only

$$g_2'(z_2)^2 = S'(i) = -1, \qquad \text{i.e.} \qquad g_2'(z_2) = \pm i,$$

both signs being compatible with the order-2 vanishing of $\tau - i$; this is exactly why the coupling through g_0 matters. Hence within the paper there is precisely one consistent choice, the printed one: $\zeta_1 = e^{-2\pi i/3}$, $\zeta_2 = e^{-2\pi i/4}$, both clockwise, which is Convention 3.17.

(S2) The deck transformation. For both j one and the same formula is used (§5),

$$\tilde{g}_j^{\log}(z, \zeta) = \big(g_j z,\ R_{g_j}(z)\zeta + \Pi(g_j z)\, v_j/m_j\big),$$

i.e. $(z, x) \mapsto (g_j z, A_j x + v_j/m_j)$ in flat coordinates, with translation part $+v_j/m_j$; and $(\tilde{g}_j^{\log})^{m_j} = t_{v_j}$ (Proposition 5.6), because $\sum_{i<m_j} A_j^i(v_j/m_j) = v_j$ for $v_j \in \Lambda^{A_j}$ — the same computation for both j.

(S3) Regluing and the zero section. The regluing is by the translation $\sigma_j = +\frac{\log s_j}{2\pi i}\Pi(z)v_j$, so that the zero section of J appears in N_j-coordinates as $\varsigma_j = -\sigma_j = -\frac{\log s_j}{2\pi i}\Pi(z)v_j$; it is $\tilde{g}_j^{\log}$-invariant (Lemma 6.5(ii)) precisely because $\log s_j \circ g_j = \log s_j - 2\pi i/m_j$, i.e. because $\zeta_j = e^{-2\pi i/m_j}$ by (S1):

$$R_{g_j}\varsigma_j(z) + \Pi(g_j z)\tfrac{v_j}{m_j} = -\tfrac{\log s_j(z)}{2\pi i}\Pi(g_j z)v_j + \tfrac{1}{m_j}\Pi(g_j z)v_j = -\tfrac{\log s_j(z) - 2\pi i/m_j}{2\pi i}\Pi(g_j z)v_j = \varsigma_j(g_j z).$$

Had ζ_j been $e^{+2\pi i/m_j}$ for one j, this identity would fail for that j with the printed σ_j — one would need σ_j of the opposite sign there — and the drift below would flip for that j alone; this is the mechanism by which (S1) propagates. The drift is the one computed in Lemma 6.5(ii): along $s_j = \varrho e^{i\theta}$, $\theta : 0 \to 2\pi$ (so that t_j runs m_j times counterclockwise), ς_j changes by $-\Pi(z)v_j$, i.e. the section drifts by $-v_j$ in $H_1 = \Lambda$ — for both j.

(S4) From the drift to the relation. In $U = A_0^{(j)} \times \Delta_{s_j} \to N_j$ the disc $\{x_0\} \times \Delta_{s_j}$ is a normal disc to S_j, so its boundary — the loop "s_j once around counterclockwise at constant flat coordinate x_0" — is trivial in $\pi_1(N_j)$. By (S3) the zero-section lift of the m_j-fold counterclockwise meridian equals (constant-x_0 loop)$\cdot t_{-v_j}$; hence $\hat{\rho}_{j,\mathrm{ccw}}^{m_j} = t_{-v_j}$, and for the clockwise meridian of Convention 3.17

$$\hat{\rho}_j^{m_j} = t_{+v_j} \qquad (j = 1, 2), \qquad \text{i.e.} \qquad \varepsilon_1 = \varepsilon_2 = +1.$$

Equivalently, in the phrasing used above: through $z \mapsto z^{m_j}$ the clockwise base meridian lifts to a path along which s_j turns by $-2\pi/m_j$, ending at $\tilde{g}_j^{\log}(x_0, z_0)$ because $\tilde{g}_j^{\log}$ multiplies s_j by $\zeta_j = e^{-2\pi i/m_j}$; so $\hat{\rho}_j \mapsto \hat{g}_j^{+1}$ and $\hat{\rho}_j^{m_j} \mapsto \hat{g}_j^{m_j} = t_{v_j}$ — the exponent $+1$ being the same for both j precisely because ζ_1 and ζ_2 are *both* of the form $e^{-2\pi i/m_j}$, by (S1).

From $\varepsilon_1 = \varepsilon_2$ to $|p|$. With $x = \hat{\rho}_1$, $y = \hat{\rho}_2$ and c the image of $\hat{\gamma}$, the relations imposed in Theorem 7.17 read $x^3 = c^{\gamma(\varepsilon_1 v_1)}$, $y^4 = c^{\gamma(\varepsilon_2 v_2)}$, $xy = c^{\ell_0}$, whence

$$|\pi_1(X)| = \left|\det\begin{pmatrix} 3 & -\varepsilon_1\gamma(v_1) \\ 4 & \varepsilon_2\gamma(v_2) - 4\ell_0 \end{pmatrix}\right| = \big|12\ell_0 - 4\varepsilon_1\gamma(v_1) - 3\varepsilon_2\gamma(v_2)\big|.$$

For the tautological gluing at p_0 one has $\ell_0 = 0$, and with $\varepsilon_1 = \varepsilon_2 =: \varepsilon$ this is $|\varepsilon| \cdot |4\gamma(v_1) + 3\gamma(v_2)| = |4\gamma(v_1) + 3\gamma(v_2)|$, independently of every orientation convention (reversing the global Convention 3.17 replaces (ℓ_0, ℓ_1, ℓ_2) by $(-\ell_0, -\ell_1, -\ell_2)$ and leaves $|p|$ unchanged; no convention changes one of $\varepsilon_1, \varepsilon_2$ without the other, (S2)–(S4) being one formula and (S1) coupling the two rotation senses through $g_1g_2g_0 = 1$). Since $\gamma(\varepsilon) = \gamma(\varepsilon') = 1$ (§2), the data $(v_1, v_2) = (\varepsilon, -\varepsilon')$ give $(\gamma(v_1), \gamma(v_2)) = (1, -1)$ and $|p| = |4 - 3| = 1$, while $(v_1, v_2) = (\varepsilon, +\varepsilon')$ gives $(1, 1)$ and $|p| = 7$. Had one instead $\varepsilon_1 = -\varepsilon_2$, the two roles would simply swap;

this is excluded by the above. $\square$

Theorem 7.17. Let (v_1, v_2) be admissible, $\ell_j = \gamma(v_j)$, ℓ_0 as above. Then
$$\pi_1(X) = \langle c, x, y \mid c \text{ central},\ xy = c^{\ell_0},\ x^3 = c^{\ell_1},\ y^4 = c^{\ell_2} \rangle,$$
this group is abelian, and $\pi_1(X) \cong \mathbb{Z}/|p|$ with $p := 12\ell_0 - 4\ell_1 - 3\ell_2$; moreover $\gcd(p, 12) = 1$, in particular $p \neq 0$. For $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$ one has $p = -1$ and $\pi_1(X) = 1$; for comparison, the threefold X' with $v_2 = +\varepsilon'$ has $(\ell_0, \ell_1, \ell_2) = (0, 1, 1)$, $p = -7$ and $\pi_1(X') \cong \mathbb{Z}/7$.

Proof. Van Kampen applied to $X = J \cup N_0 \cup N_1 \cup N_2$ (thickened; all intersections are connected, being T^4-bundles over annuli). By Lemma 7.4 and (7.10), attaching N_j imposes $\hat{\rho}_j^{m_j} = t_{v_j}$ $(j = 1, 2)$, and attaching N_0 imposes $\lambda = 1$ for $\lambda \in \Lambda_{\text{tor}}$ together with $\tilde{a}_0 = 1$, i.e. $\hat{\rho}_1\hat{\rho}_2 = t_\mu$. Thus
$$\pi_1(X) = \langle \Lambda, \hat{\rho}_1, \hat{\rho}_2 \mid \hat{\rho}_j \lambda \hat{\rho}_j^{-1} = A_j\lambda,\ \hat{\rho}_j^{m_j} = t_{v_j},\ \hat{\rho}_1\hat{\rho}_2 = t_\mu,\ \Lambda_{\text{tor}} = 1 \rangle.$$
The normal closure of Λ_{tor} under $\langle A_1, A_2 \rangle$ is $\ker \gamma = \langle \hat{u}, \hat{w}, \hat{\delta} \rangle$ (by $A_1\hat{w} = \hat{u} - \hat{w}$), so the image of Λ is $\mathbb{Z}c$, c the image of $\hat{\gamma}$, central because γ is $\langle T_1, T_2 \rangle$-invariant; with $x := \hat{\rho}_1$, $y := \hat{\rho}_2$ this is the displayed presentation.

From $y = x^{-1}c^{\ell_0}$ the group is generated by the commuting x, c, hence abelian, with relation matrix in the basis (x, c)
$$R = \begin{pmatrix} 3 & -\ell_1 \\ 4 & \ell_2 - 4\ell_0 \end{pmatrix}, \qquad \det R = -p$$
(second row: $x^4 = c^{4\ell_0 - \ell_2}$). The first elementary divisor is $\gcd(3, 4, \dots) = 1$, so the group is $\mathbb{Z}/|p|$. Finally $p \equiv -(4\ell_1 + 3\ell_2)$ (12) with $3 \nmid 4\ell_1 + 3\ell_2$ (as $3 \nmid \ell_1$) and $2 \nmid 4\ell_1 + 3\ell_2$ (as ℓ_2 is odd), so $\gcd(p, 12) = 1$. $\square$

7.6 Integral homology by Mayer–Vietoris

We use $X = N_0' \cup X^\circ$, $N_0' \cap X^\circ = M$, and set $K_k := \ker\big(H_k(M) \to H_k(N_0') = H_k(W)\big)$.

Lemma 7.18. $H_k(M) \to H_k(W)$ is surjective for all k, and with $j : M \hookrightarrow X^\circ$ there are short exact sequences
$$0 \to H_k(X^\circ)/j_*K_k \to H_k(X) \to \ker\big(j_*|_{K_{k-1}}\big) \to 0.$$
Moreover $K_1 = \mathbb{Z}[\tilde{a}_0]$ and $K_2 = \langle \xi_{\hat{w}}, \xi_{\hat{\delta}} \rangle$, where for $\nu \in \Lambda_{\text{tor}}$
$$\xi_\nu := a_*\big([S^1_\nu] \times [\tilde{a}_0]\big) \in H_2(M)$$
is the 2-torus swept by S^1_ν along the *toric* meridian $\tilde{a}_0$ of Lemma 7.4 (a the fibrewise translation action).

Proof. The composite $H_k(F) \to H_k(M) \to H_k(W)$ is c_*, surjective by Proposition 7.12; hence the second map is surjective and Mayer–Vietoris for (N_0', X°) splits as displayed. By the Wang sequence of $M \to S^1$, $H_k(M)$ is an extension of $\ker(M_0 - I)|_{\bigwedge^{k-1}\Lambda}$ by $\operatorname{coker}(M_0 - I)|_{\bigwedge^k \Lambda}$, so $H_*(M)$ is free of ranks $(1, 3, 6, 6, 3, 1)$; against $b_*(W) = (1, 2, 4, 2, 1)$ this gives $\operatorname{rk} K_k = 1, 2, 4, 2$ for $k = 1, 2, 3, 4$.

$[\tilde{a}_0]$ maps to the generator of $\ker(M_0 - I)|_{H_0(F)} = \mathbb{Z}$, hence spans a saturated rank-one subgroup, and it bounds in N_0' since the disc $\{x = x_0,\ |t| \leq \eta\} \subset Y_{\leq \eta}$ descends (§4); so $K_1 = \mathbb{Z}[\tilde{a}_0]$.

For K_2, note first that the classes ξ_ν depend on the chosen lift of the base circle: replacing $\tilde{a}_0$ by $\tilde{a}_0 t_\lambda$ changes ξ_ν by the image of $\nu \wedge \lambda \in \bigwedge^2 \Lambda = H_2(F)$, i.e. by an element of $\operatorname{coker}(\bigwedge^2 M_0 - I)$. We always use the toric meridian. A direct computation from $M_0\hat{\gamma} = \hat{\gamma} - \hat{\delta}$, $M_0\hat{u} = \hat{u} + \hat{w}$, $M_0\hat{w} = \hat{w}$, $M_0\hat{\delta} = \hat{\delta}$ gives
$$(\textstyle\bigwedge^2 M_0 - I) \bigwedge^2 \Lambda = \langle \hat{w} \wedge \hat{\delta},\ \hat{\gamma} \wedge \hat{w} + \hat{u} \wedge \hat{\delta} \rangle,$$
so in particular $\hat{w} \wedge \hat{\delta} \in \operatorname{im}(\wedge^2 M_0 - I)$: the fibre torus T_f is null-homologous already in M (a fortiori in N_0').

Now work in the toric chart $U_{\sigma_0} \cong \mathbb{C}^3$ of $Y_{\mathfrak{F}}$ attached to the triangle
$$\sigma_0 = \operatorname{cone}\big((0, 1), (e_1, 1), (e_2, 1)\big),$$
i.e. we take $v = 0$: this is what makes the identification of the boundary torus below *literal*, the second generator being then the toric meridian on the nose (for a general v it would be the toric meridian composed

with the translation t_v, which changes the class only by an element of $\mathrm{im}(\bigwedge^2 M_0 - I)$). With $t = z_1 z_2 z_3$, consider the solid torus

$$\Sigma_1 := \{|z_1| \le \rho,\ |z_2| = r,\ z_3 = c\}, \qquad \rho\, r\, |c| = \eta,$$

so that $|t| = |z_1|\, r\, |c| \le \eta$ on Σ_1, with equality exactly on $\partial\Sigma_1 = \{|z_1| = \rho,\ |z_2| = r,\ z_3 = c\}$. Thus Σ_1 lies in $Y_{\le\eta}$ and descends into N_0', with $\partial\Sigma_1 \subset \{|t| = \eta\}$ descending into M. Its boundary is the 2-torus spanned by $(0,1)$ and $(e_1, 1)$, i.e. by $\hat{w} = e_1$ and the toric meridian: this is $\xi_{\hat{w}}$, and it bounds in N_0'; the pair $(0,1), (e_2, 1)$ gives $\xi_{\hat{\delta}}$ in the same way. Under the Wang surjection $H_2(M) \to \ker(M_0 - I)|_\Lambda = \Lambda_{\mathrm{tor}}$ one has $\xi_v \mapsto v$, so $\langle \xi_{\hat{w}}, \xi_{\hat{\delta}} \rangle$ is saturated of rank $2 = \mathrm{rk}\, K_2$. □

Cover D' by two closed discs D_1', D_2' with $D_j' \cap \Sigma = \{p_j\}$ and $\{|t_j| \le \eta\} \subset \mathrm{int}\, D_j'$, meeting in a band disjoint from Σ, and put $X_j := f^{-1}(D_j')$, so that

$$X^\circ = X_1 \cup X_2, \qquad X_1 \cap X_2 = f^{-1}(\text{band}) \simeq F,$$

the band being contractible and missing Σ. Since f is a fibre bundle over $D_j' \setminus \{p_j\} \supset D_j' \setminus \{|t_j| < \eta\}$ (Lemma 7.4(i)), the radial retraction of D_j' onto $\{|t_j| \le \eta\}$ lifts to a deformation retraction of X_j onto the tube N_j', hence onto S_j by Lemma 7.3(i); so $H_*(X_j) \cong H_*(S_j)$. Put

$$\alpha_k : \textstyle\bigwedge^k \Lambda = H_k(F) \to H_k(S_1) \oplus H_k(S_2), \qquad x \mapsto (\pi_{1*}x, -\pi_{2*}x).$$

Lemma 7.19. Let (v_1, v_2) be admissible. Then:

(a) $H_1(X^\circ) = \mathrm{coker}\, \alpha_1 \cong \mathbb{Z}$, generated by c (the common image of $\hat{\gamma}$) and $\bar{g}_1, \bar{g}_2$ with $3\bar{g}_1 = \ell_1 c$, $4\bar{g}_2 = \ell_2 c$; an isomorphism is $\Phi_1 : (c, \bar{g}_1, \bar{g}_2) \mapsto (12, 4\ell_1, 3\ell_2)$, and $j_*[\tilde{a}_0] = \ell_0 c - \bar{g}_1 - \bar{g}_2$, so $\Phi_1(j_*[\tilde{a}_0]) = p$.

(b) $\ker \alpha_1 = \mathbb{Z} v_0$, $v_0 := -\hat{u} - \hat{w} + \hat{\delta}$.

(c) $\mathrm{coker}\, \alpha_2 \cong \mathbb{Z}$, and a generator of $\mathrm{Hom}(\mathrm{coker}\, \alpha_2, \mathbb{Z})$ is $\Phi := \langle \mathcal{A}, \cdot \rangle$ for a class $\mathcal{A} \in H^2(X^\circ)$ restricting to $2q$ on F.

(d) $H_5(X^\circ) = 0$.

Proof. (a) $H_1(S_j)$ is given by Lemma 7.13. The composite

$$\Lambda \to \Lambda_{A_1} \oplus \Lambda_{A_2} \cong \mathbb{Z}^4, \qquad \lambda \mapsto (\gamma\lambda, \psi_1\lambda, -\gamma\lambda, -\psi_2\lambda)$$

has image $\langle (1,0,-1,0), (0,1,0,0), (0,0,0,1) \rangle$ (use $\hat{u} - \hat{w} \mapsto (0,1,0,0)$, $\hat{w} \mapsto (0,1,0,-1)$), with cokernel $\mathbb{Z}$ generated by c, and $m_j \bar{g}_j = [v_j]$ reads $3\bar{g}_1 = \ell_1 c$, $4\bar{g}_2 = \ell_2 c$. The relation matrix $\left(\begin{smallmatrix} -\ell_1 & 3 & 0 \\ -\ell_2 & 0 & 4 \end{smallmatrix}\right)$ has 2×2 minors $3\ell_2, -4\ell_1, 12$ of gcd 1 by admissibility, so the quotient is $\mathbb{Z}$; and $\Phi_1 = (12, 4\ell_1, 3\ell_2)$ kills both relations and is *onto* because $\gcd(12, 4\ell_1, 3\ell_2) = 1$: $3 \nmid \ell_1$ gives $\gcd(12, 4\ell_1) = 4$, and ℓ_2 odd gives $\gcd(4, 3\ell_2) = 1$. Finally $\tilde{a}_0 = t_\mu \hat{\rho}_2^{-1} \hat{\rho}_1^{-1}$ gives $j_*[\tilde{a}_0] = \ell_0 c - \bar{g}_1 - \bar{g}_2$ and $\Phi_1(j_*[\tilde{a}_0]) = 12\ell_0 - 4\ell_1 - 3\ell_2 = p$.

(b) $\Lambda_{A_j} \hookrightarrow H_1(S_j)$ is injective, so $\ker \alpha_1 = (A_1 - I)\Lambda \cap (A_2 - I)\Lambda = \{a = 0,\ 2b + c + 3d = 0 = b + c + 2d\} = \mathbb{Z} \cdot (0, -1, -1, 1)$ by Lemma 7.13.

(c) For the perfect pairings, α_2 is the transpose of $\psi : H^2(S_1) \oplus H^2(S_2) \to \bigwedge^2 V$, $(a, b) \mapsto \pi_1^* a - \pi_2^* b$; hence by Remark 7.15

$$\mathrm{coker}\, \alpha_2 \cong \mathrm{Ext}(\mathrm{coker}\, \psi, \mathbb{Z}) \oplus \mathrm{Hom}(\ker \psi, \mathbb{Z}).$$

By Lemma A.4 (which computes the sublattices themselves), $L_1 := \pi_1^* H^2(S_1) = \langle \gamma\varepsilon_1, q \rangle$ and $L_2 := \pi_2^* H^2(S_2) = \{a\gamma\varepsilon_2 + bq : a \equiv b\ (2)\}$. That Lemma is stated there for $v_2 = -\varepsilon'$; its proof uses v_2 only through the differential

$$d_2(\gamma \wedge \eta_2) = \ell_2[\eta_2] - 2c_2[\gamma] \ne 0,$$

written in the notation of Lemma A.4, where η_2 is the class fixed by (A.6) (Lemma A.1) and c_2 denotes the $\hat{\delta}$-coefficient c of v_2 of Proposition 5.6, so that $\eta_2(v_2) = 2c_2$; the non-vanishing holds for every admissible v_2 because ℓ_2 is odd, so the description of L_2 is valid for every admissible pair. Then $\ker \psi \cong L_1 \cap L_2 \subseteq (\bigwedge^2 V)^G = \mathbb{Z}q$, and $q \notin L_2$, $2q \in L_2$, so $\ker \psi \cong \mathbb{Z}$, generated by the pair (a, b) with $\pi_1^* a = \pi_2^* b = 2q$. Also $L_1 + L_2 = \langle \gamma\varepsilon_1, \gamma\varepsilon_2, q \rangle$, spanned in the basis $(\gamma u, \gamma w, \gamma\delta, uw, u\delta, w\delta)$ by

$(1,0,1,0,0,0), (0,1,1,0,0,0), (0,0,6,1,0,0)$, saturated (minor on columns $1,2,4$ equals 1); so $\operatorname{coker}\psi$ is torsion free and $\operatorname{coker}\alpha_2 \cong \mathbb{Z}$. By Mayer–Vietoris (a,b) comes from $\mathcal{A} \in H^2(X^\circ)$ with $\mathcal{A}|_F = 2q$, whose functional on $\operatorname{coker}\alpha_2 \subseteq H_2(X^\circ)$ generates the dual.

(d) In the Mayer–Vietoris sequence of $X^\circ = X_1 \cup X_2$ the term $H_5(X_1) \oplus H_5(X_2) = H_5(S_1) \oplus H_5(S_2) = 0$ vanishes (S_j being a closed 4-manifold, Lemma 7.13), and $\alpha_4 : [F] \mapsto (3[S_1], -4[S_2])$ is injective; hence $H_5(X^\circ) = 0$. $\square$

Remark 7.20. Concretely, in the basis (θ_j, η_j) of $H_2(S_j)$ with intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ coming from the Gysin sequence of the free $S^1_{\hat\delta}$-action, α_2 has columns

$$(2,-4,-2,3),\ (1,-2,-2,3),\ (3,0,-4,0),\ (0,1,0,-1),\ 0,\ 0,$$

of rank 3, with two 3×3 minors 2 and -3, confirming $\operatorname{coker}\alpha_2 \cong \mathbb{Z}$; $\Phi = (4,2,3,2)$ annihilates all columns, and $\pi_{2*}H_2(F) = \langle 2\theta_2, \eta_2\rangle$ has index 2.

Lemma 7.21 (sweeping)**.** Let S^1 act smoothly and freely on a smooth manifold Z (possibly with boundary), so that $p : Z \to Z/S^1$ is a smooth principal S^1-bundle, oriented along its fibres by the orbit direction, i.e. by the standard generator $[S^1] \in H_1(S^1)$, and let $p_! : H^{k+1}(Z) \to H^k(Z/S^1)$ be integration over the fibre for that orientation. Normalise the slant product by

$$\big\langle [S^1]\backslash\alpha,\ y\big\rangle = \big\langle \alpha,\ [S^1]\times y\big\rangle \qquad \big(\alpha \in H^{k+1}(S^1\times Z),\ y \in H_k(Z)\big).$$

With $a : S^1 \times Z \to Z$ the action map, define

$$\mathrm{sw} : H_k(Z) \to H_{k+1}(Z), \qquad \mathrm{sw}(y) := a_*\big([S^1]\times y\big).$$

Then $\langle \mathcal{A}, \mathrm{sw}(y)\rangle = \langle \iota\mathcal{A}, y\rangle$ for $\mathcal{A} \in H^{k+1}(Z)$, where $\iota\mathcal{A} := p^*p_!\mathcal{A}$. Moreover the fibrewise translation by $\hat\delta \in \Lambda^{\langle A_1, A_2\rangle}$ defines such a smooth free S^1-action on the compact manifold with boundary X°, whose restriction to F is translation, so that $\iota|_{H^*(F)} = \iota_{\hat\delta}$, contraction with $\hat\delta$. (The opposite choice of generator $[S^1]$ reverses the fibre orientation and changes ι by a global sign, immaterial below.)

Proof. First, by naturality and the above normalisation of the slant product,

$$\langle \mathcal{A}, a_*([S^1]\times y)\rangle = \langle a^*\mathcal{A}, [S^1]\times y\rangle = \langle [S^1]\backslash a^*\mathcal{A},\ y\rangle.$$

Next, $[S^1]\backslash a^*\mathcal{A} = p^*p_!\mathcal{A}$. Indeed, for a free action the map $S^1 \times Z \to Z \times_{Z/S^1} Z$, $(\theta, z) \mapsto (\theta z, z)$, is a diffeomorphism, so (pr_2, a) identifies the trivial circle bundle $\mathrm{pr}_2 : S^1 \times Z \to Z$ with the pullback of the oriented principal circle bundle $p : Z \to Z/S^1$ along p, the map a covering p; the fibre orientations correspond (both are the orbit direction, oriented by $[S^1]$). Integration over the fibre commutes with such base change, whence $(\mathrm{pr}_2)_!\, a^*\mathcal{A} = p^*p_!\mathcal{A}$; and for the product bundle pr_2, integration over the fibre is, with the above normalisation of the slant product and of the fibre orientation, precisely $[S^1]\backslash(\cdot)$. This proves the first assertion.

Since $\hat\delta$ is $\langle A_1, A_2\rangle$-invariant and the transition maps R_g are linear, translation by $t\hat\delta$ ($t \in \mathbb{R}/\mathbb{Z}$) is well defined and smooth on J and on N_1, N_2. On J it is translation by $t\hat\delta$ in the fibre torus $(\Lambda\otimes\mathbb{R})/\Lambda$, free because $\hat\delta$ is primitive. On N_j it suffices to check freeness on $S_j = A_0^{(j)}/\langle\alpha\rangle$, $\alpha(x) = A_jx + v_j/m_j$: the image of x is fixed by $t\hat\delta$ iff

$$(A_j^k - I)x + \big(\textstyle\sum_{i<k} A_j^i\big)\frac{v_j}{m_j} + t\hat\delta \in \Lambda \quad \text{for some } 0 \le k < m_j.$$

Applying the A_j-invariant form γ, which kills $\hat\delta$, gives $k\ell_j/m_j \in \mathbb{Z}$; as $\gcd(\ell_j, m_j) = 1$ by admissibility this forces $k = 0$, and then $t\hat\delta \in \Lambda$, i.e. $t = 0$. The same computation applies on the nearby fibres of N_j, so the action is smooth and free on X°, which is a compact manifold with boundary M.

Finally, the inclusion $i : F \hookrightarrow X^\circ$ of a smooth fibre is S^1-equivariant, and both a_* and the slant product are natural, so $i^*(\iota\mathcal{A}) = \iota_F(i^*\mathcal{A})$, where $\iota_F = p_F^*p_{F!}$ is the corresponding operator for the translation action on F. On the torus $F = (\Lambda\otimes\mathbb{R})/\Lambda$ this action is that of the subcircle $\mathbb{R}\hat\delta/\mathbb{Z}\hat\delta$; $p_F : F \to F/S^1_{\hat\delta}$ is a torus quotient, $H^*(F) = \bigwedge^* V$ is computed by translation-invariant forms, p_F^* is the inclusion of $\bigwedge^*(\hat\delta^\perp)$, and integration of an invariant form over the (oriented) orbit circles is contraction with $\hat\delta$. Hence $\iota_F = p_F^*p_{F!} = \iota_{\hat\delta}$ on $H^*(F)$. For the fixed points of the finite subgroups $\mu_n \subset S^1_{\hat\delta}$ see Remark 9.25. $\square$

Theorem 7.22. For every admissible (v_1, v_2) and every ℓ_0, with $p = 12\ell_0 - 4\ell_1 - 3\ell_2$,
$$H_*(X;\mathbb{Z}) = \big(\mathbb{Z},\ \mathbb{Z}/p,\ \mathbb{Z}/p,\ \mathbb{Z}/p,\ \mathbb{Z}/p,\ 0,\ \mathbb{Z}\big), \qquad e(X) = e(W) = 2.$$
In particular $(\ell_0, \ell_1, \ell_2) = (0, 1, -1)$ gives $|p| = 1$ and $H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z})$, while for comparison the threefold X' with $v_2 = +\varepsilon'$ has all groups in degrees $1, \dots, 4$ equal to $\mathbb{Z}/7$.

Proof. Degree 1. $H_1(X) = \pi_1(X)^{\mathrm{ab}} = \mathbb{Z}/|p|$ by Theorem 7.17; consistently, Lemma 7.18 gives $H_1(X) = H_1(X^\circ)/j_*K_1 = \mathbb{Z}/\Phi_1(j_*[\tilde{a}_0])$ and $\ker(j_*|_{K_1}) = 0$ since $p \neq 0$.

Degree 2. Hence $H_2(X) = H_2(X^\circ)/j_*K_2$. Mayer–Vietoris for $X^\circ = X_1 \cup X_2$ gives
$$0 \to \operatorname{coker}\alpha_2 \to H_2(X^\circ) \xrightarrow{\partial} \ker\alpha_1 \to 0, \qquad \operatorname{coker}\alpha_2 \cong \mathbb{Z},\ \ker\alpha_1 = \mathbb{Z}\nu_0.$$
For $\nu \in \Lambda_{\mathrm{tor}}$ the torus ξ_ν is cut by $X_1 \cap X_2 \simeq F$ into two cylinders, and transporting them gives
$$\partial(j_*\xi_\nu) = \pm(I - A_1^{-1})\nu = \pm(I - A_2)\nu$$
(the two agree on Λ_{tor} because $A_1A_2 = M_0^{-1}$ acts trivially there). Since $A_2\hat{w} = -\hat{u} + \hat{\delta}$, $\partial(j_*\xi_{\hat{w}}) = \mp\nu_0$ generates $\ker\alpha_1$, while $\partial(j_*\xi_{\hat{\delta}}) = 0$; so $j_*\xi_{\hat{\delta}} \in \operatorname{coker}\alpha_2$, and since $j_*\xi_{\hat{w}}$ maps onto a generator of $\ker\alpha_1$, the quotient
$$H_2(X) = H_2(X^\circ)/j_*K_2 = \operatorname{coker}\alpha_2 \big/ \mathbb{Z}\, j_*\xi_{\hat{\delta}}$$
is *cyclic*, of order $|\Phi(j_*\xi_{\hat{\delta}})|$, $\operatorname{coker}\alpha_2 \cong \mathbb{Z}$ being infinite cyclic with Φ generating its dual (Lemma 7.19(c)). Now $\xi_{\hat{\delta}} = \mathrm{sw}([\tilde{a}_0])$, so $\Phi(j_*\xi_{\hat{\delta}}) = \langle \iota\mathcal{A}, j_*[\tilde{a}_0]\rangle$ by Lemma 7.21. The class $\iota\mathcal{A} \in H^1(X^\circ)$ restricts on F to
$$\iota_{\hat{\delta}}(2q) = \mp 12\gamma$$
(with $q = uw + 6\gamma\delta$: $\iota_{\hat{\delta}}(uw) = 0$ and $\iota_{\hat{\delta}}(\gamma \wedge \delta) = \mp\gamma$, the sign depending only on the convention for contracting a 2-form, and being immaterial below since Φ is anyhow defined up to sign). But $H^1(X^\circ) = \operatorname{Hom}(H_1(X^\circ), \mathbb{Z}) = \mathbb{Z}\Theta$ with $\Theta = \Phi_1$ and $\Theta|_F = 12\gamma$ (since $\lambda \mapsto \gamma(\lambda)c$ on H_1). As restriction to F is injective on $H^1(X^\circ)$, $\iota\mathcal{A} = \mp\Theta$ and
$$\Phi(j_*\xi_{\hat{\delta}}) = \mp\Phi_1(j_*[\tilde{a}_0]) = \mp p,$$
so $H_2(X) \cong \mathbb{Z}/|p|$ (cyclic by the preceding display) and $j_*|_{K_2}$ is injective.

Degrees 3, 4, 5. $H_1(X), H_2(X)$ are finite, so $b_1 = b_2 = 0$, and Poincaré duality with universal coefficients on the closed oriented 6-manifold X gives
$$\begin{aligned} H^1(X) &= \operatorname{Hom}(H_1, \mathbb{Z}) = 0, \\ H_5(X) &\cong H^1(X) = 0, \\ H_4(X) &\cong H^2(X) \cong \operatorname{Ext}(H_1, \mathbb{Z}) = \mathbb{Z}/p. \end{aligned}$$
Moreover $X \setminus W \to B \setminus \{p_0\}$ is proper with all fibres of Euler number 0 (smooth 2-tori and the bielliptic S_1, S_2), so $e(X) = e(W) = 2$ by Proposition 7.11; then $2 = 1 + 0 + 0 - b_3 + 0 + 0 + 1$ gives $b_3 = 0$ and $H_3(X) \cong H^3(X) \cong \operatorname{Ext}(H_2, \mathbb{Z}) = \mathbb{Z}/p$. ☐

Remark 7.23. Degree 4 also follows from Lemma 7.18: K_4 is generated by the 3-tori $T_{\hat{\gamma}\hat{w}\hat{\delta}}, T_{\hat{u}\hat{w}\hat{\delta}}$ swept along the toric meridian, $\partial(T_{\hat{\gamma}\hat{w}\hat{\delta}})$ generates $\ker\alpha_3$, and pairing the second generator against $\operatorname{coker}\alpha_4 = (\mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2])/(3, -4) \xrightarrow{(4,3)} \mathbb{Z}$ gives again $\mp p$.

7.7 A second computation: the integral Leray spectral sequence

Remark 7.24. $H_*(X;\mathbb{Z})$ is computed twice. The second computation — Theorem 7.17 (with Lemma 7.16), §7.4, §7.7 and Appendix B, together with Corollary 4.8 — (for admissible pairs with $4 \mid \varepsilon_2(v_2)$, in particular for X) does not use §7.2 (Proposition 7.2) or Proposition 7.11; §7.2 is used by the first computation (§§7.3–7.6), for the explicit collapse map c and the degree-2 normalisations of Proposition 7.12, and, through Proposition 7.12 and Lemma A.2(6), in §9 (Proposition 9.13, Lemma 9.12(iv)); the recognition of X as S^6 (Theorem 7.17, Corollary 7.29, §8) does not use it. The description of $H_*(W)$ and of the collapse in the Main Theorem still rests on §7.2.

We recompute $H^*(X;\mathbb{Z})$ from $E_2^{a,b} = H^a(B, R^b f_*\mathbb{Z})$, independently of §7.6. Near p_0 this subsection uses exactly one thing: that $R^q f_*\mathbb{Z} \to j_*j^*R^q f_*\mathbb{Z}$ is an isomorphism near p_0, i.e. that $\mathrm{sp}^q\colon H^q(W;\mathbb{Z}) = (R^q f_*\mathbb{Z})_{p_0} \to H^q(F;\mathbb{Z})$ is injective with image $(\wedge^q V)^{T_0}$. This is Theorem B.1 of Appendix B, proved there from the toric local models by nearby cycles, without the retraction of §7.2 (it is also the invariants statement of Proposition 7.12). The other inputs — the local data at p_1, p_2 (Lemma 7.13, Proposition 7.14), the meridian relations (7.10) of §7.5, and Lemma 7.25 together with the Mayer–Vietoris computation of $\ker f^*$ inside Proposition 7.27 (both proved in this subsection) — are independent of the Mayer–Vietoris computation of §7.6, so that the conclusion $H^2(X;\mathbb{Z}) \cong H^3(X;\mathbb{Z}) \cong \mathbb{Z}/|p|$, and with it $H_2(X;\mathbb{Z}) = 0$ when $|p| = 1$ (by universal coefficients: $\mathrm{Hom}(H_2,\mathbb{Z})$ is a summand of $H^2(X) = 0$ and $\mathrm{Ext}(H_2,\mathbb{Z})$ a summand of $H^3(X) = 0$), is obtained without any appeal to §7.6. By proper base change $(R^q f_*\mathbb{Z})_p = H^q(f^{-1}(p);\mathbb{Z})$; the specialisation map at p_0 is the sheaf map sp^q of Appendix B, and at p_j it is π_j^*, since N_j' retracts radially onto S_j (Lemma 7.3(i) for N_j, which does not involve §7.2). Write $\mathcal{L}^q := \bigwedge^q R^1 f_*\mathbb{Z}|_{B^\circ}$, $j : B^\circ \hookrightarrow B$, $G := \langle T_1, T_2\rangle$, and $H^k(R^q) := H^k(B, R^q f_*\mathbb{Z})$. In this subsection (v_1, v_2) denotes an admissible pair with $4 \mid \varepsilon_2(v_2)$ — in particular the pair $(\varepsilon, -\varepsilon')$ of §6, for which $\varepsilon_2(-\varepsilon') = 0$ —, this being the hypothesis under which Proposition 7.14 identifies the degree-3 cokernel at p_2; Propositions 7.26 and 7.27 are stated for such pairs.

Lemma 7.25. Let $\mathcal{L}$ be a local system of free finitely generated $\mathbb{Z}$-modules on $B^\circ = B \setminus \Sigma$, with monodromy group G acting on the stalk M.

(i) $H^2(B, j_*\mathcal{L}) \cong M_G$, the coinvariants.

(ii) These isomorphisms are compatible with cup products: if $\mathcal{L}', \mathcal{L}''$ are two such local systems with stalks M', M'' and $\mathcal{L}' \otimes \mathcal{L}''$ has stalk $M' \otimes M''$, then under $H^0(B, j_*\mathcal{L}') = (M')^G$, $H^2(B, j_*\mathcal{L}'') \cong (M'')_G$ and $H^2(B, j_*(\mathcal{L}' \otimes \mathcal{L}'')) \cong (M' \otimes M'')_G$ the cup product is $x \smile [y] = [x \otimes y]$, and symmetrically for $H^2 \otimes H^0$.

Proof. (i) From $0 \to j_!\mathcal{L} \to j_*\mathcal{L} \to \bigoplus_{p\in\Sigma} M^{G_p} \to 0$ (skyscrapers, which have no higher cohomology) one gets $H^2(B, j_!\mathcal{L}) \cong H^2(B, j_*\mathcal{L})$, and $H^2(B, j_!\mathcal{L}) = H^2_c(B^\circ, \mathcal{L}) \cong H_0(B^\circ, \mathcal{L}) = M_G$ by Poincaré duality with local coefficients on the oriented surface B° [Hat02, §3.H] (see also [Bre97, Ch. V]).

(ii) Each step of the composite in (i) — restriction to B°, the duality isomorphism $H^2_c(B^\circ, \mathcal{L}) \cong H_0(B^\circ, \mathcal{L})$, and the identification $H_0(B^\circ, \mathcal{L}) = M_G$ — is natural in $\mathcal{L}$ and compatible with coefficient pairings. Concretely, the duality isomorphism is cap product with the fundamental class $[B^\circ] \in H_2^{\mathrm{lf}}(B^\circ)$, and cap product with a fixed class is natural in the coefficient system and satisfies the projection formula

$$(x \smile \alpha) \frown [B^\circ] = x \frown \big(\alpha \frown [B^\circ]\big), \qquad x \in H^0(B^\circ, \mathcal{L}'),\ \alpha \in H^2_c(B^\circ, \mathcal{L}''),$$

the right-hand operation being the coefficient action $y \mapsto x \otimes y$ on H_0 (for cup/cap products with coefficient pairings and for Poincaré duality with local coefficients see [Hat02, §3.H], [Bre97, Ch. V]). Passing to $H_0(B^\circ, \mathcal{L}'') = (M'')_G$, the action of a G-invariant $x \in M'$ is $[y] \mapsto [x \otimes y]$, which is well defined precisely because x is G-invariant. This is the asserted compatibility. □

Proposition 7.26. $R^q f_*\mathbb{Z}$ is the subsheaf of $j_*\mathcal{L}^q$ determined by the three specialisation lattices; near p_0 the inclusion is an equality for all q, and the quotient Q^q is the skyscraper

$$\begin{aligned} &Q^1 = \mathbb{Z}/3 \oplus \mathbb{Z}/4 \quad (\text{at } p_1, p_2,\ \text{generated by } [\gamma]),\\ &Q^2 = \mathbb{Z}/2, \quad Q^3 = \mathbb{Z}/2 \quad (\text{at } p_2,\ \text{generated by } [q],\ [\gamma u w]),\\ &Q^4 = \mathbb{Z}/3 \oplus \mathbb{Z}/4. \end{aligned}$$

Consequently

$$\begin{aligned} &H^0(B, R^1) = 12\mathbb{Z}\gamma, && H^0(B, R^2) = \mathbb{Z}\cdot 2q,\\ &H^0(B, R^3) = \mathbb{Z}\cdot 2\gamma u w, && H^0(B, R^4) = 12\mathbb{Z}\cdot \mathrm{vol},\\ &H^2(B, R^q) = (\textstyle\bigwedge^q V)_G = \mathbb{Z}[\bar\delta],\ \mathbb{Z}[\gamma\delta],\ \mathbb{Z}[uw\delta],\ \mathbb{Z} \quad (q = 1,2,3,4), \end{aligned}$$

and $H^1(B,R^1) = H^1(B,R^2) = 0$. The isomorphisms $H^2(B,R^q) \cong (\bigwedge^q V)_G$ are compatible with cup products in the sense of Lemma 7.25(ii) — that is, they are induced by the naturality of Poincaré duality $H^2_c \cong H_0$ with respect to coefficient pairings; this is what is used in Proposition 7.27.

Proof. The sheaves and cokernels are described by Theorem B.1 (at p_0; in particular $Q^q_{p_0} = 0$) and Proposition 7.14 (at p_1, p_2); the generator $[\gamma u w]$ of Q^3 is the one of Proposition 7.14, stated there under the hypothesis $4 \mid \varepsilon_2(v_2)$, which holds for our $v_2 = -\varepsilon'$. From $0 \to R^q \to j_*\mathcal{L}^q \to Q^q \to 0$,

$$0 \to H^0(R^q) \to (\textstyle\bigwedge^q V)^G \to \oplus_i Q^q_i \to H^1(R^q) \to H^1(j_*\mathcal{L}^q) \to 0,$$

and $H^2(R^q) \cong H^2(B, j_*\mathcal{L}^q) \cong (\bigwedge^q V)_G$ by Lemma 7.25(i), the skyscraper quotient having no H^2; the cup-product compatibility is Lemma 7.25(ii), the inclusion $R^q \subset j_*\mathcal{L}^q$ being a map of sheaves of rings up to the evident pairings. Here $(\bigwedge^q V)^G = \mathbb{Z}\gamma,\ \mathbb{Z}q,\ \mathbb{Z}\gamma u w,\ \mathbb{Z}\,\mathrm{vol}$, and the coinvariants are $V_G = \mathbb{Z}\bar{\delta}$ (as $(T_1 - I)V + (T_2 - I)V = \langle \gamma, u, w\rangle$), $(\bigwedge^2 V)_G = \mathbb{Z}[\gamma\delta]$ with $[uw] = 6[\gamma\delta]$ (the sum of the images is the saturated lattice $\langle \gamma u, \gamma w, u\delta, w\delta, uw - 6\gamma\delta\rangle$), $(\bigwedge^3 V)_G = \mathbb{Z}[uw\delta]$; all torsion free. For $q = 1$ the map $\mathbb{Z}\gamma \to \mathbb{Z}/3 \oplus \mathbb{Z}/4$ is onto with kernel $12\mathbb{Z}\gamma$, for $q = 2$ the map $\mathbb{Z}q \to \mathbb{Z}/2$ is onto; hence $H^1(R^q) \cong H^1(B, j_*\mathcal{L}^q) =: K^q$ (parabolic cohomology). Normalising a cocycle on $\pi_1(B^\circ) = F(t_1, t_2)$ so that $c(t_2) = 0$ (possible as $c(t_2) \in (T_2 - I)M$), with residual freedom M^{T_2}, and using $c(t_0) = -T_0 c(t_1)$, one finds

$$K^q \cong \frac{(T_1 - I)M \cap (T_0 - I)M}{(T_1 - I)M^{T_2}}, \qquad M = \textstyle\bigwedge^q V.$$

For $q = 1$: $(T_1 - I)V = \langle 2\gamma + u + w,\ 2\gamma - u\rangle$ and $(T_0 - I)V = \langle \gamma, u\rangle$ meet in $\mathbb{Z}(2\gamma - u) = \mathbb{Z}(T_1 - I)\varepsilon_2$, so $K^1 = 0$. For $q = 2$: $(\bigwedge^2 T_1 - I)\bigwedge^2 V = \langle \gamma u, \gamma w,\ 2u\delta + w\delta,\ uw + u\delta - w\delta - 6\gamma\delta\rangle$ while $(\bigwedge^2 T_0 - I)\bigwedge^2 V = \langle \gamma u, \gamma w + u\delta\rangle$; an element $a\gamma u + b(\gamma w + u\delta)$ lies in the former only if $b = 0$ (compare the $uw, u\delta, w\delta$ coefficients), so the intersection is $\mathbb{Z}\gamma u = \mathbb{Z}(\bigwedge^2 T_1 - I)(\gamma\varepsilon_2)$ and $K^2 = 0$. ☐

Proposition 7.27. $d_2^{0,1}(12\gamma) = \pm p\,\omega$, $d_2^{0,2}(2q) = \pm p\,\omega\bar{\delta}$, $d_2^{0,3}(2\gamma u w) = \pm p\,\omega[\gamma\delta]$, with one and the same sign in the three formulas and ω generating $H^2(B;\mathbb{Z})$. Consequently

$$H^1(X;\mathbb{Z}) = 0, \qquad H^2(X;\mathbb{Z}) \cong H^3(X;\mathbb{Z}) \cong \mathbb{Z}/|p|,$$

in agreement with Theorems 7.17 and 7.22.

Proof. $d_2^{0,1}(12\gamma)$ is the obstruction to lifting the section 12γ to $H^1(X;\mathbb{Z})$. Over N_0 a lift exists since $\gamma \in V^{T_0}$ (Theorem B.1, together with the Leray sequence of f over D_0, $H^2(D_0; f_*\mathbb{Z}) = 0$); over N_j one exists since $12\gamma \in \pi_j^* H^1(S_j)$ (Proposition 7.14), and any lift Θ_j satisfies $m_j\Theta_j(\hat{g}_j) = 12\ell_j$, i.e. $\Theta_j(\hat{\rho}_j) = 12\ell_j/m_j$, equal to $4\ell_1, 3\ell_2$, while $\Theta_0(\tilde{a}_0) = 0$ because $\tilde{a}_0$ is null-homotopic in N_0 (Corollary 4.8(ii)). Evaluating the discrepancy on $\tilde{a}_0 = t_\mu \hat{\rho}_2^{-1}\hat{\rho}_1^{-1}$ gives $d_2^{0,1}(12\gamma) = \pm(12\ell_0 - 4\ell_1 - 3\ell_2)\omega$.

Two remarks on this computation. First, $E_2^{2,0} = H^2(B;\mathbb{Z}) = \mathbb{Z}\omega$ and, since only $d_2^{0,\bullet}$ can be nonzero, $E_\infty^{2,0} = E_3^{2,0} = \operatorname{coker} d_2^{0,1} = \operatorname{im}\big(f^* : H^2(B) \to H^2(X)\big)$; equivalently

$$\operatorname{im} d_2^{0,1} = \ker\big(f^* : H^2(B;\mathbb{Z}) \to H^2(X;\mathbb{Z})\big).$$

Second, this kernel is computed by Mayer–Vietoris, as follows (no Čech computation on a cover of B by non-acyclic pieces is involved). Let $U := D_0 \sqcup D_1 \sqcup D_2$ be the disjoint union of small open discs around p_0, p_1, p_2 and let V be a neighbourhood of $B \setminus \bigcup_i \overline{D_i'}$ (with $D_i' \Subset D_i$), so that $B = U \cup V$, $U \cap V$ is a disjoint union of three annuli and V is homotopy equivalent to a pair of pants. Since $H^1(U) = 0$ and $H^2(U) = H^2(V) = 0$, Mayer–Vietoris gives an isomorphism

$$H^1(U \cap V)/\operatorname{im} H^1(V) \xrightarrow{\ \sim\ } H^2(B;\mathbb{Z}) = \mathbb{Z}\omega;$$

evaluating on the three (clockwise) meridians identifies $H^1(U \cap V) = \mathbb{Z}^3$, and $H^1(V) \to \mathbb{Z}^3$ is injective with image the sum-zero sublattice (the three boundary meridians of the pair of pants compose to 1), so the class of (n_0, n_1, n_2) is $\pm(n_0 + n_1 + n_2)\omega$. Now the section 12γ lifts over $f^{-1}(V)$ to a class Θ_V (possible since $H^2(V) = 0$) and over $f^{-1}(D_i)$ to the Θ_i above, and $d_2^{0,1}(12\gamma)$ is the class of the triple of differences $\Theta_i - \Theta_V$ on the three annuli. Writing $\alpha_i := \Theta_V(\hat{\rho}_i)$ and using $\tilde{a}_0 = t_\mu \hat{\rho}_2^{-1}\hat{\rho}_1^{-1}$, whence $\Theta_V(\tilde{a}_0) = 12\ell_0 - \alpha_1 - \alpha_2$, the three meridian values of the differences are

$$4\ell_1 - \alpha_1, \qquad 3\ell_2 - \alpha_2, \qquad -(12\ell_0 - \alpha_1 - \alpha_2),$$

of sum $-(12\ell_0 - 4\ell_1 - 3\ell_2) = -p$; hence $d_2^{0,1}(12\gamma) = \pm p\,\omega$, as computed above.

For the other differentials use multiplicativity of the Leray spectral sequence, together with the cup-product description of the E_2-terms in Proposition 7.26 (Lemma 7.25(ii), i.e. the naturality of the duality $H^2_c \cong H_0$ with respect to coefficient pairings); the generator $2\gamma uw$ of $H^0(B, R^3)$ is the one provided there, the hypothesis $4 \mid \varepsilon_2(v_2)$ of Proposition 7.14 being satisfied for our $v_2 = -\varepsilon'$. Normalise the sign by $d_2^{0,1}(12\gamma) = p\,\omega$ and write $d_2(2q) = p'\omega\bar{\delta}$, $d_2(2\gamma uw) = p''\omega[\gamma\delta]$. From $(12\gamma)(2q) = 12(2\gamma uw)$, the Leibniz rule with the sign $(-1)^{\deg} = -1$ of the total degree 1 of 12γ, and $[q] = 12[\gamma\delta]$, one gets

$$12p'' = 24p - 12p', \qquad \text{i.e.} \qquad p'' = 2p - p'.$$

From $(2q)(2\gamma uw) \in E_2^{0,5} = 0$ together with the pairings $E_2^{2,1} \times E_2^{0,3} \to E_2^{2,4} \cong \mathbb{Z}$ and $E_2^{0,2} \times E_2^{2,2} \to E_2^{2,4}$, both ± 2 on generators, one gets $|p'| = |p''|$. The case $p'' = -p'$ would give $p = 0$, excluded; hence $p' = p'' = p$, which is the assertion (with the common sign).

Since $E_2^{a,b} = 0$ for $a \geq 3$, only $d_2^{0,\bullet}$ can be nonzero, so $H^1(X) = \ker d_2^{0,1} = 0$ and

$$H^2(X): \ E_\infty^{2,0} = \mathbb{Z}/p, \quad E_\infty^{1,1} = 0, \quad E_\infty^{0,2} = \ker(\times p) = 0,$$
$$H^3(X): \ E_\infty^{2,1} = V_G/pV_G = \mathbb{Z}/p, \quad E_\infty^{1,2} = E_\infty^{0,3} = 0.$$

□

Remark 7.28. Both routes single out the same arithmetic: $p \equiv -(4\ell_1 + 3\ell_2)$ (12) is local data at p_1, p_2 (admissibility being exactly smoothness of X at the multiple fibres), while ℓ_0 is the free gluing parameter at p_0; thus $|p| = 1$ iff $(\ell_1 \bmod 3, \ell_2 \bmod 4) \in \{(1,3), (2,1)\}$, realised by $(0,1,-1)$ and not by $(0,1,1)$.

Corollary 7.29. For the threefold X of §6 ($v_1 = \varepsilon$, $v_2 = -\varepsilon'$, tautological gluing at p_0) one has $(\ell_0, \ell_1, \ell_2) = (0,1,-1)$, $p = -1$, and

$$\pi_1(X) = 1, \qquad H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z}), \qquad e(X) = c_3(X)[X] = 2.$$

Proof. $\pi_1(X) = 1$ is Theorem 7.17 with $|p| = 1$. By Propositions 7.26 and 7.27, $H^1(X;\mathbb{Z}) = 0$ and $H^2(X;\mathbb{Z}) \cong H^3(X;\mathbb{Z}) \cong \mathbb{Z}/|p| = 0$. By universal coefficients $\mathrm{Hom}(H_2(X), \mathbb{Z}) \subset H^2(X) = 0$ and $\mathrm{Ext}(H_2(X), \mathbb{Z}) \subset H^3(X) = 0$, so $H_2(X) = 0$; $H_1(X) = \pi_1(X)^{\mathrm{ab}} = 0$. By Poincaré duality on the closed oriented 6-manifold X, $H_4(X) \cong H^2(X) = 0$ and $H_5(X) \cong H^1(X) = 0$; the rank of $H_3(X)$ is that of $H^3(X) = 0$ and its torsion is that of $H^4(X) \cong H_2(X) = 0$, so $H_3(X) = 0$. Hence $H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z})$ and $e(X) = 2$; finally $c_3(X)[X] = e(X)$, since the top Chern class of T_X is the Euler class of the underlying oriented real vector bundle [Hir, §4.10]. This argument uses §7.4, §7.5 (with Lemma 7.4), §7.7, Appendix B and Corollary 4.8 only — in particular not §7.2, Proposition 7.11 or §7.6; Theorem 7.22 gives the same groups by the first route. □

8 Recognition: X is diffeomorphic to S^6.

Throughout this section X denotes the compact complex threefold constructed in §6, regarded as a closed smooth 6-manifold and oriented by its complex structure.

Theorem 8.1. X is diffeomorphic to S^6.

Lemma 8.2. Let M be a closed smooth simply connected n-manifold, $n \geq 2$, with $\widetilde{H}_*(M;\mathbb{Z}) \cong \widetilde{H}_*(S^n;\mathbb{Z})$. Then M is homotopy equivalent to S^n.

Proof. By the Hurewicz theorem [Hat02, Thm. 4.32], applied inductively in degrees $2, \ldots, n$, one gets $\pi_i(M) = 0$ for $i \leq n-1$ and an isomorphism $\pi_n(M) \xrightarrow{\sim} H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. Let $\varphi\colon S^n \to M$ represent a class mapping to a generator. By naturality of the Hurewicz map φ_* is an isomorphism on H_n, hence (the remaining reduced groups vanishing) on all integral homology. Both spaces are simply connected and of the homotopy type of CW complexes: this is clear for S^n, and for the closed topological manifold M it is [Hat02, Cor. A.12]. Hence φ is a homotopy equivalence by the homology Whitehead theorem [Hat02, Cor. 4.33]. □

Proof of Theorem 8.1. By Theorem 6.2, X is a compact complex manifold of dimension 3, so its underlying smooth manifold is closed of real dimension 6; by Theorem 7.17 and Corollary 7.29 (equivalently Theorem 7.22), $\pi_1(X) = 1$ and $H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z})$. Lemma 8.2 with $n = 6$ shows that X is a homotopy 6-sphere. By Smale's generalised Poincaré theorem in dimensions ≥ 5 [Sma61, Thm. A], X is homeomorphic to S^6 and is a twisted sphere. By the h-cobordism theorem [Mil65], for $n \geq 5$ the h-cobordism classes of oriented homotopy n-spheres coincide with their oriented diffeomorphism classes, and by Kervaire–Milnor [KM63, Lem. 4.5, Thm. 1.1] these classes form a finite abelian group Θ_n under connected sum, with $\Theta_6 = 0$ [KM63, §7]. Hence X is orientation-preservingly diffeomorphic to S^6 with one of its two orientations; as S^6 carries an orientation-reversing self-diffeomorphism (a reflection of $\mathbb{R}^7$), X is diffeomorphic to S^6, and the diffeomorphism may be taken orientation-preserving for the complex orientation. $\square$

Remark 8.3. The proof uses only $\pi_1(X) = 1$ (Theorem 7.17) and $H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z})$ (Corollary 7.29 or Theorem 7.22); $H_*(X;\mathbb{Z})$ is computed twice in §7: once by the Mayer–Vietoris route, which uses the retraction and collapse of §7.2, and once by the Leray route of §7.7 with Appendix B, which together with Theorem 7.17 and Corollary 4.8 does not use §7.2; the recognition of X as S^6 therefore does not depend on Proposition 7.2.

Remark 8.4. It is the diffeomorphism, not merely the homeomorphism of [Sma61], that is needed for the statement of §1: transporting the complex structure of X through any diffeomorphism $X \to S^6$ yields an integrable almost complex structure on the *standard* smooth S^6 whose complex manifold is biholomorphic to X.

Corollary 8.5. $e(X) = 2$ and $c_3(X) = \langle c_3(T_X), [X]\rangle = 2$, where $[X]$ is the fundamental class for the complex orientation.

Proof. From Corollary 7.29 (or Theorem 7.22), $e(X) = \sum(-1)^i b_i(X) = 2$; equivalently $e(X) = e(W) + 0 + 0$ as in §7, the multiple fibres $3S_1, 4S_2$ contributing $e(S_j) = 0$ (S_j bielliptic, Theorem 5.4) and the remaining fibres of f being 2-tori. For a compact complex n-fold, $c_n(T_X)$ is the Euler class of the underlying oriented real tangent bundle, so $\langle c_n(T_X), [X]\rangle = e(X)$ [Hir, §4.10]; take $n = 3$. $\square$

Remark 8.6. In particular $c_3(X) \neq 0$, and this non-vanishing is what rules out $a(X) = 2$ in §9, while the vanishing $b_2(X) = 0$ obtained in §7 rules out $a(X) = 3$.

9 Complex-analytic invariants of X.

Throughout, $f\colon X \to B = \mathbb{P}^1$ is the map of §6 and the notation is that of §2–§5: $\Sigma = \{p_0, p_1, p_2\}$, $B^\circ = B \smallsetminus \Sigma$, $F_b = f^{-1}(b) = \mathbb{C}^2/\Pi(z)\Lambda$ for $b = \pi(z) \in B^\circ$, $W = f^{-1}(p_0)$, $f^*p_j = m_j S_j$ with S_j bielliptic. Write $\sigma_1 := 6\mu\gamma + \tau u + w$, $\sigma_2 := \beta\gamma + \mu u + \delta$ for the rows of $\Pi = [Z \mid I]$, viewed in $V \otimes \mathbb{C} = \mathrm{Hom}(\Lambda, \mathbb{C})$, so $\sigma_i(\lambda) = (\Pi\lambda)_i$ and

$$F^1(z) := \langle \sigma_1(z), \sigma_2(z)\rangle = H^0(\Omega^1_{F_{\pi(z)}}) \subset H^1(F_{\pi(z)}, \mathbb{C}).$$

Put $e := d\zeta_1 \wedge d\zeta_2$, the flat relative 2-form on $\mathcal{T}$. As usual $\mathcal{M}(Y)$ denotes the field of meromorphic functions of a compact complex space Y, and $a(Y) = \mathrm{trdeg}_{\mathbb{C}}\, \mathcal{M}(Y)$.

Theorem 9.1. Let X and f be as in §6. Then:

(i) $a(X) = 1$, $\mathcal{M}(X) = f^*\mathcal{M}(B)$, f is the algebraic reduction of X, and the *very general* fibre of f has algebraic dimension 0;

(ii) $f_*\mathcal{O}_X = \mathcal{O}_B$, $R^1 f_*\mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-1)$, $R^2 f_*\mathcal{O}_X \cong \mathcal{O}(-1)$; hence $h^{0,0} = h^{0,1} = 1$, $h^{0,2} = h^{0,3} = 0$;

(iii) $f_*\omega_{X/B} \cong \mathcal{O}_B(1)$ and $K_X \cong f^*\mathcal{O}(-1) \otimes \mathcal{O}(2S_2)$; in particular K_X is not a torsion line bundle;

(iv) $c_1(X) = c_2(X) = 0$ in $H^*(X;\mathbb{Z})$; $c_1c_2 = 0$, $c_3 = e(X) = 2$, $\chi(\mathcal{O}_X) = 0$, $\chi(\Omega^1_X) = -1$, $\chi(T_X) = 1$;

(v) $h^0(T_X) = 1$ and $\mathrm{Aut}^0(X) = \mathbb{C}^*$, acting fibrewise, with fixed locus one of the three double curves of W (a smooth rational curve through both triple points) and normal weights $(+1, -1)$;

(vi) $H^0(X, \Omega^p_X) = 0$ for $p = 1, 2, 3$, i.e.

$$h^{1,0} = h^{2,0} = h^{3,0} = 0$$

(the case $p = 3$ being consistent with $h^{0,3} = 0$ in (ii) by Serre duality); together with (ii), (iv) and Serre duality this determines every Hodge number of X in terms of the single integer $h^{1,2} = h^{2,1}$, namely $h^{1,1} = h^{2,2} = h^{1,2} + 1$ and $h^{p,0} = h^{0,q} = 0$ for $p \geq 1$, $q \geq 2$.

Parts (i)–(vi) are proved in §§9.1–9.7. The Hodge number $h^{1,2} = h^{2,1}$ is not computed here; see Remark 9.21.

9.1 The algebraic dimension

In this subsection we prove Theorem 9.1(i). The argument rests on the period description of the smooth fibres (§2, §3) and on elementary facts about complex 2-tori; neither §7 nor [CDP20] enters (see Remark 9.9).

Notation. Write $\bigwedge^\bullet V$ for the exterior algebra of $V = \mathrm{Hom}(\Lambda, \mathbb{Z})$, let $(\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta})$ be the basis of Λ dual to the basis (γ, u, w, δ) of V, and put

$$\mathrm{vol} := \gamma \wedge u \wedge w \wedge \delta \in \textstyle\bigwedge^4 V \cong \mathbb{Z}.$$

For $E, E' \in \bigwedge^2 V$ write $E \wedge E' = (E \cdot E')\,\mathrm{vol}$ and $E^2 := E \cdot E$; under the marking of §3, $\bigwedge^2 V = H^2(F_b; \mathbb{Z})$ for $b = \pi(z) \in B^\circ$, the product $(\,\cdot\,)$ being the cup product evaluated on $\pm$ the fundamental class (only non-vanishing statements will be used, so the orientation is immaterial). A class $E \in \bigwedge^2 V$ is written

$$E = E_{\gamma u}\,\gamma \wedge u + E_{\gamma w}\,\gamma \wedge w + E_{\gamma\delta}\,\gamma \wedge \delta + E_{uw}\,u \wedge w + E_{u\delta}\,u \wedge \delta + E_{w\delta}\,w \wedge \delta, \qquad E_{ij} \in \mathbb{Z}.$$

Finally set

$$\eta := u \wedge w + 6\,\gamma \wedge \delta \ \in\ \textstyle\bigwedge^2 V.$$

We shall use the following specialisations to the generators g_0, g_2 of Δ of the laws $(\tau 2)$, $(\mu 1)$, $(\beta 1)$ of §2, valid on all of $\mathfrak{h}_z$:

$$\begin{aligned} &\tau \circ g_0 = \tau - 1, \quad \mu \circ g_0 = \mu, \qquad\quad\ \beta \circ g_0 = \beta + 1, \\ &\tau \circ g_2 = -1/\tau, \quad \mu \circ g_2 = 1 + \mu/\tau, \quad \beta \circ g_2 = \beta - 3 - 6\mu^2/\tau. \end{aligned} \tag{9.1}$$

Lemma 9.2 (Néron–Severi groups of the smooth fibres)**.** Let $z \in \mathfrak{h}_z^\circ$, $b = \pi(z)$, and $E \in \bigwedge^2 V = H^2(F_b; \mathbb{Z})$. Then

$$E \in \mathrm{NS}(F_b) := H^2(F_b; \mathbb{Z}) \cap H^{1,1}(F_b) \iff P_E(z) = 0,$$

where P_E is the holomorphic function on $\mathfrak{h}_z$

$$P_E := E_{\gamma u} - E_{\gamma w}\,\tau - E_{\gamma\delta}\,\mu + 6E_{uw}\,\mu + E_{u\delta}\,\beta + E_{w\delta}\,(6\mu^2 - \tau\beta).$$

Proof. For a complex torus $H^2(F_b; \mathbb{C}) = \bigwedge^2(V \otimes \mathbb{C})$ and $H^{p,q} = \bigwedge^p F^1 \wedge \bigwedge^q \overline{F^1}$; in particular $H^{2,0} = \mathbb{C}\,\sigma_1 \wedge \sigma_2$ and $H^{0,2} = \mathbb{C}\,\bar{\sigma}_1 \wedge \bar{\sigma}_2$. Since $L(z) = \Pi(z)\Lambda$ is a lattice of full rank (Theorem 3.4), $F^1 \oplus \overline{F^1} = V \otimes \mathbb{C}$, so $\sigma_1, \sigma_2, \bar{\sigma}_1, \bar{\sigma}_2$ is a basis of $V \otimes \mathbb{C}$ and

$$\bar{\sigma}_1 \wedge \bar{\sigma}_2 \wedge \sigma_1 \wedge \sigma_2 \neq 0 \quad \text{in } \textstyle\bigwedge^4(V \otimes \mathbb{C}).$$

Let E be real (e.g. integral) and write $E = E^{2,0} + E^{1,1} + E^{0,2}$. As $\dim F^1 = 2$ we have $\bigwedge^3 F^1 = 0$, so $E^{2,0} \wedge \sigma_1 \wedge \sigma_2 = 0$ and $E^{1,1} \wedge \sigma_1 \wedge \sigma_2 = 0$; writing $E^{0,2} = c\,\bar{\sigma}_1 \wedge \bar{\sigma}_2$ we get

$$E \wedge \sigma_1 \wedge \sigma_2 = c\,\bar{\sigma}_1 \wedge \bar{\sigma}_2 \wedge \sigma_1 \wedge \sigma_2.$$

Hence E is of type $(1,1)$ if and only if $E \wedge \sigma_1 \wedge \sigma_2 = 0$.

Now compute, from $\sigma_1 = 6\mu\gamma + \tau u + w$ and $\sigma_2 = \beta\gamma + \mu u + \delta$,

$$\sigma_1 \wedge \sigma_2 = (6\mu^2 - \tau\beta)\,\gamma \wedge u - \beta\,\gamma \wedge w + 6\mu\,\gamma \wedge \delta - \mu\,u \wedge w + \tau\,u \wedge \delta + w \wedge \delta.$$

The non-zero products of complementary basis bivectors are

$$\begin{aligned} &\gamma u \wedge w\delta = \mathrm{vol}, \quad \gamma w \wedge u\delta = -\mathrm{vol}, \quad \gamma\delta \wedge uw = \mathrm{vol}, \\ &uw \wedge \gamma\delta = \mathrm{vol}, \quad u\delta \wedge \gamma w = -\mathrm{vol}, \quad w\delta \wedge \gamma u = \mathrm{vol}, \end{aligned}$$

(all other products of two distinct basis bivectors vanish), whence

$$E \wedge \sigma_1 \wedge \sigma_2 = \Big(E_{\gamma u} \cdot 1 - E_{\gamma w} \cdot \tau - E_{\gamma\delta} \cdot \mu + E_{uw} \cdot 6\mu + E_{u\delta} \cdot \beta + E_{w\delta}(6\mu^2 - \tau\beta)\Big)\mathrm{vol} = P_E(z)\,\mathrm{vol}. \qquad \square$$

Lemma 9.3 (linear independence of the period functions)**.** If $a, b, c, d, e \in \mathbb{C}$ satisfy

$$a + b\,\tau + c\,\mu + d\,\beta + e\,(6\mu^2 - \tau\beta) \equiv 0 \quad \text{on } \mathfrak{h}_z,$$

then $a = b = c = d = e = 0$. Consequently, for $E \in \bigwedge^2 V$,

$$P_E \equiv 0 \iff E \in \mathbb{Z}\eta.$$

Proof. By (9.1),

$$(6\mu^2 - \tau\beta) \circ g_0 = 6\mu^2 - (\tau - 1)(\beta + 1) = (6\mu^2 - \tau\beta) + (\beta - \tau + 1).$$

Evaluating the relation at $g_0 z$ and subtracting the relation at z gives

$$-b + d + e\,(\beta - \tau + 1) \equiv 0.$$

If $e \neq 0$ this makes $\beta - \tau$ constant, which is impossible because $(\beta - \tau) \circ g_0 = \beta - \tau + 2$. Hence $e = 0$ and $d = b$, and the relation becomes

$$a + b(\tau + \beta) + c\mu \equiv 0.$$

By (9.1), $(\tau + \beta) \circ g_2 = -1/\tau + \beta - 3 - 6\mu^2/\tau$ and $\mu \circ g_2 = 1 + \mu/\tau$; evaluating at $g_2 z$, subtracting, and multiplying by $-\tau$ we obtain

$$6b\mu^2 + c(\tau - 1)\mu + b(\tau^2 + 3\tau + 1) - c\tau \equiv 0. \tag{9.2}$$

Evaluating (9.2) at $g_0 z$ and subtracting (9.2) (use $\mu \circ g_0 = \mu$, $\tau \circ g_0 = \tau - 1$, and $(\tau - 1)^2 + 3(\tau - 1) + 1 = \tau^2 + 3\tau + 1 - 2(\tau + 1)$) gives

$$c\mu + 2b(\tau + 1) - c \equiv 0.$$

If $c \neq 0$ this reads $\mu = 1 - 2b(\tau + 1)/c$; applying g_0 and using $\mu \circ g_0 = \mu$, $\tau \circ g_0 = \tau - 1$ yields $2b/c = 0$, i.e. $b = 0$. If $c = 0$ it gives $b = 0$ directly. So in both cases $b = 0$, and (9.2) becomes $c\,(\mu\tau - \mu - \tau) \equiv 0$. If $c \neq 0$ then $\mu = \tau/(\tau - 1)$, while applying g_0 gives $\mu = (\tau - 1)/(\tau - 2)$; equating, $\tau(\tau - 2) = (\tau - 1)^2$, i.e. $0 = 1$. Hence $c = 0$, and then $a = 0$; finally $d = b = 0$.

For the consequence: by Lemma 9.2,

$$P_E = E_{\gamma u} + (-E_{\gamma w})\tau + (6E_{uw} - E_{\gamma\delta})\mu + E_{u\delta}\beta + E_{w\delta}(6\mu^2 - \tau\beta),$$

so $P_E \equiv 0$ if and only if $E_{\gamma u} = E_{\gamma w} = E_{u\delta} = E_{w\delta} = 0$ and $E_{\gamma\delta} = 6E_{uw}$, i.e. if and only if $E = E_{uw}\,(u \wedge w + 6\,\gamma \wedge \delta) = E_{uw}\,\eta$. $\square$

Lemma 9.4 (η is an indefinite $(1,1)$-class)**.** For every $z \in \mathfrak{h}_z^\circ$ one has $\eta \in \mathrm{NS}(F_{\pi(z)})$, $\eta^2 = 12$, and the hermitian form H_η of η on $\mathbb{C}^2$ is non-degenerate of signature $(1,1)$. Consequently, for every integer $n \neq 0$,

$$(n\eta)^2 = 12n^2 \neq 0, \qquad H_{n\eta} = n\,H_\eta \text{ is non-degenerate and indefinite.}$$

Proof. $(1,1)$. For $E = \eta$ one has $E_{uw} = 1$, $E_{\gamma\delta} = 6$ and all other $E_{ij} = 0$, so $P_\eta = -6\mu + 6\mu \equiv 0$ and Lemma 9.2 applies. (This is the Lagrangian relation $Z_{11} = 6\mu$ of §2.)

Self-intersection. $\eta \wedge \eta = 2 \cdot 6\,(u \wedge w) \wedge (\gamma \wedge \delta) = 12\,\mathrm{vol}$, since $u \wedge w \wedge \gamma \wedge \delta = \gamma \wedge u \wedge w \wedge \delta$.

Signature. Regard η as an alternating form on Λ: $\eta(\hat{u}, \hat{w}) = 1$, $\eta(\hat{\gamma}, \hat{\delta}) = 6$, and η vanishes on the remaining pairs of basis vectors. Its inverse $\eta^\vee$ is the alternating form on V with

$$\eta^\vee(u, w) = -1, \qquad \eta^\vee(\gamma, \delta) = -\tfrac{1}{6},$$

vanishing on the remaining pairs.

We first record the exact relation between the hermitian form and the Gram matrix. By Appell–Humbert and the Riemann bilinear relations (see e.g. [BL04, Ch. 2–4]), a class $E \in \mathrm{NS}$ is the imaginary part of a unique hermitian form H_E on $\mathbb{C}^2$; write $H_E(\zeta, \zeta') = \sum_{a,b}(H_E)_{ab}\zeta_a\bar{\zeta}'_b$ in the linear coordinates ζ_1, ζ_2 on $\mathbb{C}^2$ dual to σ_1, σ_2, so that $d\zeta_a = \sigma_a$ in $V \otimes \mathbb{C}$. Put

$$G_E := \big(i\,E^\vee(\sigma_a, \bar{\sigma}_b)\big)_{a,b=1,2}, \qquad E^\vee \text{ the inverse of } E.$$

Then, as a 2-form, $\operatorname{Im} H_E = \frac{1}{2i}\sum_{a,b}(H_E)_{ab}\, d\zeta_a \wedge d\bar{\zeta}_b$, i.e.

$$E = -\tfrac{i}{2}\sum_{a,b}(H_E)_{ab}\,\sigma_a \wedge \bar{\sigma}_b;$$

hence in the basis $(\sigma_1, \sigma_2, \bar{\sigma}_1, \bar{\sigma}_2)$ of $V \otimes \mathbb{C}$ the matrix of E is $\left(\begin{smallmatrix} 0 & A \\ -A^{\mathsf{T}} & 0 \end{smallmatrix}\right)$ with $A = -\frac{i}{2}H_E$, whose inverse is $\left(\begin{smallmatrix} 0 & -(A^{\mathsf{T}})^{-1} \\ A^{-1} & 0 \end{smallmatrix}\right)$, so that $E^{\vee}(\sigma_a, \bar{\sigma}_b) = -2i\,\overline{(H_E^{-1})}_{ab}$. Therefore

$$G_E = 2\,\overline{H_E^{-1}}, \qquad \text{equivalently} \qquad H_E = 2\,\overline{G_E}^{\,-1};$$

in particular H_E and G_E are simultaneously non-degenerate and have the same signature.

A direct computation with $\sigma_1 = 6\mu\gamma + \tau u + w$, $\sigma_2 = \beta\gamma + \mu u + \delta$ gives

$$\eta^{\vee}(\sigma_1, \bar{\sigma}_1) = -\tau + \bar{\tau} = -2i\operatorname{Im}\tau, \quad \eta^{\vee}(\sigma_1, \bar{\sigma}_2) = -\mu + \bar{\mu} = -2i\operatorname{Im}\mu, \quad \eta^{\vee}(\sigma_2, \bar{\sigma}_2) = -\tfrac{\beta - \bar{\beta}}{6} = -\tfrac{i}{3}\operatorname{Im}\beta,$$

so that

$$G_\eta = \begin{pmatrix} 2\operatorname{Im}\tau & 2\operatorname{Im}\mu \\ 2\operatorname{Im}\mu & \frac{1}{3}\operatorname{Im}\beta \end{pmatrix}, \qquad \det G_\eta = \tfrac{2}{3}\operatorname{Im}\tau\operatorname{Im}\beta - 4(\operatorname{Im}\mu)^2 = \tfrac{2}{3}\operatorname{Im}\tau \cdot D,$$

with $D = \operatorname{Im}\beta - 6(\operatorname{Im}\mu)^2/\operatorname{Im}\tau$. By Theorem 3.4, $\operatorname{Im}\tau > 0$ and $D < 0$ on $\mathfrak{h}_z$, so $\det G_\eta < 0$: the real symmetric 2×2 matrix G_η is non-degenerate with eigenvalues of opposite signs, i.e. of signature $(1,1)$, and by the relation above the same holds for $H_\eta = 2G_\eta^{-1}$ (G_η being real). For instance at $\tau = iT$, $\mu = 0$, $\beta = -iT$ (so $D = -T < 0$) one finds $G_\eta = \operatorname{diag}(2T, -T/3)$ and $H_\eta = 2G_\eta^{-1} = \operatorname{diag}(1/T, -6/T)$. □

Proposition 9.5. The set

$$\mathcal{C} := \{z \in \mathfrak{h}_z^{\circ} : \ \operatorname{NS}(F_{\pi(z)}) \neq \mathbb{Z}\eta\}$$

is countable, and $a(F_{\pi(z)}) = 0$ for every $z \in \mathfrak{h}_z^{\circ} \smallsetminus \mathcal{C}$.

Proof. By Lemmas 9.2 and 9.4, $\mathbb{Z}\eta \subseteq \operatorname{NS}(F_{\pi(z)})$ for every z, and

$$\mathcal{C} = \bigcup_{E \in \bigwedge^2 V \smallsetminus \mathbb{Z}\eta} \{z \in \mathfrak{h}_z^{\circ} : \ P_E(z) = 0\}.$$

For $E \notin \mathbb{Z}\eta$ the holomorphic function P_E is not identically zero on the connected domain $\mathfrak{h}_z$ (Lemma 9.3), so its zero set is discrete, hence countable; and $\bigwedge^2 V \cong \mathbb{Z}^6$ is countable. Thus $\mathcal{C}$ is countable.

Let now $z \notin \mathcal{C}$, put $T := F_{\pi(z)}$, and suppose $a(T) \geq 1$. By the structure theory of the algebraic reduction of a complex torus ([LB99, Ch. 2]; see also [BHPV, Ch. IV,VI]) there is a subtorus $T_1 \subset T$ such that $p \colon T \to T_a := T/T_1$ is the algebraic reduction and T_a is an abelian variety of dimension $a(T)$.

- If $a(T) = 2$ then $T = T_a$ is an abelian surface; let $h \in \operatorname{NS}(T)$ be the class of an ample line bundle. Then H_h is positive definite, whereas $h = n\eta$ for some $n \neq 0$ and $H_{n\eta}$ is indefinite by Lemma 9.4: contradiction.

- If $a(T) = 1$ then T_a is an elliptic curve and T_1 is a 1-dimensional subtorus; the class $\varphi := p^*[\mathrm{pt}] \in \operatorname{NS}(T)$ of a fibre satisfies $\varphi^2 = 0$. Moreover φ is the Poincaré dual of the fibre class $[T_1] \in H_2(T;\mathbb{Z})$, so for a flat Kähler form ω on T one has

$$\int_T \varphi \wedge \omega = \int_{T_1} \omega > 0,$$

  whence $\varphi \neq 0$. But $\varphi = n\eta$ with $n \neq 0$ forces $\varphi^2 = 12n^2 \neq 0$ (Lemma 9.4): contradiction.

Hence $a(T) = 0$. □

Lemma 9.6. Let $g \in \mathcal{M}(X)$ and suppose that $g|_{F_b}$ is constant for uncountably many $b \in B^{\circ}$. Then $g = f^*\varphi$ for a (unique) $\varphi \in \mathcal{M}(B) = \mathbb{C}(t)$.

Proof. Let $|P| \subset X$ be the polar divisor of g (a hypersurface containing the indeterminacy locus of g) and put

$$\Omega := f^{-1}(B^{\circ}) \smallsetminus |P|,$$

a connected open subset of X on which g is holomorphic and f is a submersion. Only finitely many fibres are contained in $|P|$ (each is then a union of irreducible components of $|P|$); for every other $b \in B^\circ$ the set $F_b \cap \Omega$ is a connected 2-dimensional closed submanifold of Ω, dense in F_b, and $g|_{F_b}$ is a well-defined meromorphic function on F_b.

Let

$$Z := \{x \in \Omega : \ (dg \wedge f^* d\mathrm{t})(x) = 0\} \subset \Omega,$$

an analytic subset, where t denotes a local holomorphic coordinate on B (the condition is independent of the choice). If $g|_{F_b}$ is constant then $F_b \cap \Omega \subset Z$.

Assume $Z \neq \Omega$. Then Z is a proper analytic subset of the connected 3-fold Ω, so each of its irreducible components has dimension ≤ 2, and there are at most countably many components (Ω is second countable). Each $F_b \cap \Omega$ contained in Z is an irreducible closed analytic subset of Z of dimension 2, hence is contained in, and therefore equal to, one of the (finitely or countably many) 2-dimensional components; distinct b give disjoint, hence distinct, components. So only countably many b would have $g|_{F_b}$ constant, contrary to hypothesis.

Hence $Z = \Omega$, i.e. dg annihilates $\ker df$ at every point of Ω; as the sets $F_b \cap \Omega$ are connected, g is constant on each of them, and $g = \varphi \circ f$ on Ω with φ a function on B° minus the finite set of the bad b's. The function φ is holomorphic there: locally $\varphi = g \circ s$ for a holomorphic local section s of the submersion $f|_\Omega$.

It remains to see that φ is meromorphic at each of the finitely many remaining points $t_0 \in B$. Choose $x \in f^{-1}(t_0)$ and local coordinates centred at x; since the germs of $f^{-1}(t_0)$ and of $|P|$ at x are hypersurface germs, there is a holomorphic disc $\kappa\colon (\Delta, 0) \to (X, x)$ with $\kappa(\Delta) \not\subset |P|$ and $f \circ \kappa$ non-constant, say $f \circ \kappa(s) = t_0 + s^k \cdot (\text{unit})$ in a local coordinate at t_0. Then $g \circ \kappa$ is meromorphic on Δ, and $g \circ \kappa = \varphi \circ (f \circ \kappa)$ off a discrete subset of Δ. Consequently $|g \circ \kappa(s)| \leq C|s|^{-M}$ near 0 for some M, whence $|\varphi(t)| \leq C'|t - t_0|^{-M/k}$ on a punctured neighbourhood of t_0; so $(t - t_0)^N \varphi$ is bounded for $N \gg 0$ and extends holomorphically by Riemann's theorem, i.e. φ is meromorphic at t_0. Thus $\varphi \in \mathcal{M}(B) = \mathbb{C}(t)$ and $g = f^*\varphi$ on the dense open set Ω, hence everywhere. □

Proposition 9.7. $a(X) = 1$, $\mathcal{M}(X) = f^*\mathcal{M}(B)$, and f is the algebraic reduction of X.

Proof. Let $g \in \mathcal{M}(X)$ with polar divisor $|P|$. By Proposition 9.5 the set $\pi(\mathcal{C}) \subset B^\circ$ is countable, and only finitely many fibres are contained in $|P|$; since B is uncountable, there are uncountably many $b \in B^\circ$ with

$$a(F_b) = 0 \quad \text{and} \quad F_b \not\subset |P|.$$

For such b the restriction $g|_{F_b}$ is a meromorphic function on the compact connected torus F_b, hence constant. By Lemma 9.6, $g \in f^*\mathcal{M}(B)$. Therefore

$$\mathcal{M}(X) = f^*\mathcal{M}(B) \cong \mathbb{C}(t),$$

the pullback f^* being injective because f is surjective; in particular $a(X) = \operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(t) = 1$. Finally, f is a surjective holomorphic map onto the projective curve B with connected fibres and $f^*\mathcal{M}(B) = \mathcal{M}(X)$, i.e. f is the algebraic reduction of X. □

Corollary 9.8. The set of $b \in B$ for which F_b is not a smooth fibre with $\mathrm{NS}(F_b) = \mathbb{Z}\eta$ is contained in a countable union of proper analytic subsets of B (i.e. is countable); for all other b,

$$\mathrm{NS}(F_b) = \mathbb{Z}\eta \quad \text{and} \quad a(F_b) = 0.$$

In particular the *very general* fibre of the algebraic reduction f has algebraic dimension 0, and carries no non-constant meromorphic function.

Proof. Immediate from Proposition 9.5: the exceptional set is $\Sigma \cup \pi(\mathcal{C})$, a countable subset of the curve B, and proper analytic subsets of B are exactly the finite sets.

The statement is independent of the marking used to identify $H^2(F_b;\mathbb{Z})$ with $\bigwedge^2 V$: the class η is invariant under the monodromy group of the family over B° (Lemma A.1(6) of the Appendix; equivalently, the monodromy-invariance lemma of §2), so the subgroup $\mathbb{Z}\eta \subset H^2(F_b;\mathbb{Z})$, and hence the equality $\mathrm{NS}(F_b) = \mathbb{Z}\eta$, does not depend on the chosen marking. □

Remark 9.9 (second proof that $a(X) = 1$). We record the route through [CDP20]. It yields $a(X) = 1$, and hence also $\mathcal{M}(X) = f^*\mathcal{M}(B)$: indeed $\mathcal{M}(X)$ then has transcendence degree 1 over $\mathbb{C}$, so it is algebraic over the subfield $f^*\mathcal{M}(B)$, which is algebraically closed in $\mathcal{M}(X)$ because f is proper and surjective with connected fibres onto the normal curve B [Ue75, §3]; whence $\mathcal{M}(X) = f^*\mathcal{M}(B)$. It does *not* yield Corollary 9.8, and it uses the topological computations of §7 and §9.4.

(a) $a(X) \neq 3$. A Moishezon threefold has $b_2 > 0$ [CDP98, Lemma 1.4], while $b_2(X) = 0$ by §7.

(b) If $a(Y) \geq 2$ for a compact connected complex manifold Y, then some non-constant $h \in \mathcal{M}(Y)$ is not *holomorphic as a map $Y \dashrightarrow \mathbb{P}^1$.* Choose algebraically independent $g_1, g_2 \in \mathcal{M}(Y)$. If one of them is already non-holomorphic, we are done; otherwise $\Phi := (g_1, g_2)\colon Y \to \mathbb{P}^1 \times \mathbb{P}^1$ is holomorphic, and $\Phi(Y)$ is an irreducible compact analytic subset (Remmert) of dimension 2 — a point or a curve (algebraic by Chow) would give an algebraic relation between g_1 and g_2 — so Φ is surjective. By upper semicontinuity of fibre dimension, $S := \{q : \dim \Phi^{-1}(q) \geq 2\}$ is a proper analytic subset, and it is finite: if it contained a curve Γ then $\dim \Phi^{-1}(\Gamma) \geq 3 = \dim Y$, so $\Phi(Y) \subset \Gamma$. Pick $(a_1, a_2) \in \mathbb{C}^2 \setminus S$ and $x \in \Phi^{-1}(a_1, a_2)$, and put $u := g_1 - a_1$, $v := g_2 - a_2 \in \mathcal{O}_{Y,x}$: both vanish at x, and they are coprime in the factorial ring $\mathcal{O}_{Y,x}$, since a common prime factor p would give a hypersurface germ $\{p = 0\} \subset (\Phi^{-1}(a_1, a_2), x)$, which has dimension ≤ 1. Then

$$h := \frac{1}{g_1 - a_1} + \frac{1}{g_2 - a_2} = \frac{u + v}{uv}$$

is a reduced fraction at x (a prime dividing uv divides u or v, and if it also divided $u + v$ it would divide both), with both $u + v$ and uv vanishing at x; hence x is an indeterminacy point of h. (The choice $(a_1, a_2) \notin S$ is essential: if the level sets shared a component with different multiplicities, such a sum could well be holomorphic there.)

(c) $a(X) \neq 2$. If $a(X) = 2$, then by (b) there is a non-constant meromorphic, non-holomorphic map $X \dashrightarrow \mathbb{P}^1$, so the hypotheses of [CDP20, Thm. 2.1] are met. We quote that theorem verbatim [CDP20, Thm. 2.1, p. 679] (in it, B denotes a holomorphic vector bundle, not our base curve, and g is the meromorphic map, not a generator of Δ):

> *2.1. Theorem. Let X be a 3-dimensional compact complex manifold with $b_2(X) = 0$. Assume that there exists a non-holomorphic meromorphic non-constant map $g\colon X \dashrightarrow \mathbb{P}_1$. Let B be a holomorphic vector bundle on X. Then*
> *a) $H^i(X, B \otimes M) = 0$ for $i \geq 0$ and $M \in \mathrm{Pic}^\circ(X)$ generic.*
> *b) $\chi(X, B \otimes M) = 0$ for all $M \in \mathrm{Pic}^\circ(X)$.*
> *c) $c_3(X) = 0$; i.e., either $b_1(X) = 0$ and $b_3(X) = 2$, or $b_1(X) = 1$ and $b_3(X) = 0$.*

followed there by *"Another proof has been given in [LSS18]."* (loc. cit.'s [LSS18] is our [LRS]). Our X has $b_2(X) = 0$ (§7), so part c) would give $c_3(X) = 0$, contradicting $c_3(X) = e(X) = 2$ (§9.4). (See also [LRS, Thm. 3.1].) With (a) and $f^*\mathcal{M}(B) \subset \mathcal{M}(X)$ this gives $a(X) = 1$.

(d) A related holomorphic statement. Independently of the above,

$$\text{if } h\colon X \to \mathbb{P}^1 \text{ is holomorphic, then } h = \varphi \circ f \text{ for a (unique) holomorphic } \varphi\colon B \to \mathbb{P}^1. \tag{9.3}$$

Indeed, $L := h^*\mathcal{O}_{\mathbb{P}^1}(1)$ has $c_1(L) = 0$ because $H^2(X; \mathbb{Z}) = 0$ (§7); if $h|_{F_b}$ were non-constant for some $b \in B^\circ$, it would be surjective and $D := (h|_{F_b})^{-1}(\infty)$ would be a non-empty effective divisor with $\mathcal{O}_{F_b}(D) \cong L|_{F_b}$. On a compact complex torus a line bundle with vanishing first Chern class is given by a unitary character of the lattice (Appell–Humbert, [BL04, Ch. 2]), and its sections are the entire functions θ on $\mathbb{C}^2$ with $\theta(\zeta + \ell) = \chi(\ell)\theta(\zeta)$, which are bounded, hence constant, and non-zero only for $\chi \equiv 1$; so $\mathcal{O}_{F_b}(D) \cong \mathcal{O}_{F_b}$ with $D \geq 0$ forces $D = 0$, a contradiction. Thus h is constant on the fibres over B° and factors through the proper map f with connected fibres onto the normal curve B [Ue75, §3].

(e) Caveat. X contradicts [CDP20, Cor. 2.3], the failure being located in [CDP20, Thm. 2.2] (see §10); using [CDP20, Thm. 2.1] is legitimate only insofar as its proof is independent of that step (it is, and [LRS] reproves it). The proof of Proposition 9.7 uses neither §7 nor [CDP20].

9.2 Automorphy factors and the canonical bundle

Recall $[P_g \mid R_g] = \Pi(gz)M_g^{-1}$, i.e.

$$R_g(z)\,\Pi(z)\lambda = \Pi(gz)\,A_g\lambda \quad (\lambda \in \Lambda), \qquad A_g := M_g^{-1}. \tag{9.4}$$

Lemma 9.10. (i) $R_{g_0} \equiv I$, $\det R_{g_1} = -1/\tau$, $\det R_{g_2} = 1/\tau$; moreover $R_{g_j}(z_j)$ fixes $e_2 = \Pi(z_j)\hat{\delta}$ and has second eigenvalue $\zeta_j^{a_j}$, with $(a_1, a_2) = (2, 1)$; in particular $\det R_{g_j}(z_j) = \zeta_j^{a_j}$ and $0 < a_j < m_j$.

(ii) On J_j' the form $s_j^{k_j} e^{\varphi_j}\, ds_j \wedge e$, with $k_j := m_j - 1 - a_j$ (so $k_1 = 0$, $k_2 = 2$) and e^{φ_j} a suitable unit on Δ_j, is invariant under $\langle \tilde{g}_j^{\log} \rangle$ and descends to a *section* of K_{N_j} whose divisor is $k_j S_j$; it is a generator of K_{N_j} precisely where it does not vanish, i.e. everywhere if $k_j = 0$ and on $N_j \setminus S_j$ if $k_j > 0$. Consequently $K_{N_1} \cong \mathcal{O}_{N_1}$, $K_{N_2} \cong \mathcal{O}_{N_2}(2S_2)$, and $K_{N_0} \cong \mathcal{O}_{N_0}$.

Proof. (i) By (9.4) the columns of $R_g(z)$ are $\Pi(gz)A_g\hat{w}$ and $\Pi(gz)A_g\hat{\delta}$. With $A_1\hat{w} = \hat{u} - \hat{w}$, $A_1\hat{\delta} = \hat{\delta}$ and $A_2\hat{w} = -\hat{u} + \hat{\delta}$, $A_2\hat{\delta} = \hat{\delta}$ a direct computation gives, writing $\tau^\sharp = \tau(g_j z)$, $\mu^\sharp = \mu(g_j z)$,

$$R_{g_1} = \begin{pmatrix} \tau^\sharp - 1 & 0 \\ \mu^\sharp & 1 \end{pmatrix}, \qquad R_{g_2} = \begin{pmatrix} -\tau^\sharp & 0 \\ 1 - \mu^\sharp & 1 \end{pmatrix},$$

so $\det R_{g_1} = (\tau - 1)/\tau - 1 = -1/\tau$ and $\det R_{g_2} = -\tau(g_2 z) = 1/\tau$; for g_0, $M_0\hat{w} = \hat{w}$, $M_0\hat{\delta} = \hat{\delta}$ and $(\tau, \mu, \beta) \circ g_0 = (\tau - 1, \mu, \beta + 1)$ give $R_{g_0} = I$. As $\hat{\delta}$ is A_j-invariant, $R_{g_j}(z_j)e_2 = e_2$; in the basis $(e_1, e_2) = (\Pi\hat{w}, \Pi\hat{\delta})$ the matrix $R_{g_j}(z_j)$ is lower triangular with $(2,2)$-entry 1, so its second eigenvalue is its $(1,1)$-entry — the vector e_1 itself need not be an eigenvector — namely $\tau(z_1) - 1 = \rho - 1 = e^{2\pi i/3} = \zeta_1^2$ for $j = 1$ and $-\tau(z_2) = -i = \zeta_2$ for $j = 2$. Since the $(2,2)$-entry is 1, the determinant equals this second eigenvalue $\zeta_j^{a_j}$, and $(a_1, a_2) = (2, 1)$ satisfies $0 < a_j < m_j = (3, 4)_j$.

(ii) On $J_j' = (\Delta_j \times \mathbb{C}^2)/\Lambda$ the form $ds_j \wedge e$ is Λ-invariant and nowhere zero, and since $\tilde{g}_j^{\log}(z, \zeta) = (g_j z, R_{g_j}(z)\zeta + \Pi(g_j z)v_j/m_j)$ and the ds_j-components die against ds_j,

$$(\tilde{g}_j^{\log})^*\big(F\, ds_j \wedge e\big) = (F \circ g_j)\,\chi(s_j)\, ds_j \wedge e, \qquad \chi := \zeta_j \det R_{g_j}.$$

Here χ is nowhere zero on Δ_j with $\prod_{i<m_j} \chi(\zeta_j^i s) = 1$; writing $\chi = \chi(0)e^\psi$, $\psi(0) = 0$, the $\mathbb{Z}/m_j$-invariant part of ψ vanishes, so $\varphi(s) - \varphi(\zeta_j s) = \psi(s)$ is solvable in $\mathcal{O}(\Delta_j)$, and $F = s^k e^\varphi$ satisfies $F(\zeta_j s)\chi(s) = F(s)$ exactly when $\zeta_j^k \chi(0) = 1$. By (i) $\chi(0) = \zeta_j^{1+a_j}$, so $k \equiv k_j$; the holomorphic invariant forms $F\, ds_j \wedge e$ are thus exactly those with $F \in s^{k_j}e^{\varphi_j} \cdot \mathcal{O}(D_j)$, $\mathcal{O}(D_j)$ being the invariant functions. Since $\langle \tilde{g}_j^{\log} \rangle$ acts freely (§5), $J_j' \to N_j$ is étale, the invariant form descends, and its divisor is $k_j S_j$ (the reduced divisor $\{s_j = 0\}$ on J_j' being the preimage of S_j). For N_0: all rays of $\mathfrak{F}$ lie at height 1, so $t^{-1}(0) = \sum_v D_v$ is reduced and $\Omega_0 := \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge dt$ generates $K_{Y_{\mathfrak{F}}}$; it is invariant under torus translations and under the shears $\Phi_{\bar{\lambda}}$ (as $\det \varphi_{\bar{\lambda}} = 1$), hence under all $\Psi_{\bar{\lambda}}$, and descends. □

Proposition 9.11. $\mathfrak{K} := f_*\omega_{X/B}$ is invertible of degree 1. The multivalued frame e of $\mathfrak{K}|_{B^\circ}$ has automorphy factor $\det R_g$, generates $\mathfrak{K}$ at p_0, and vanishes at p_j to order a_j/m_j, i.e. $2/3$ at p_1, $1/4$ at p_2. Moreover $K_X \cong f^*\mathcal{O}(-1) \otimes \mathcal{O}(2S_2)$, and K_X is not a torsion line bundle.

Proof. $\mathfrak{K}$ is torsion free of rank 1 on the smooth curve B, hence invertible; e is Λ-invariant with $\tilde{g}^*e = \det R_g \cdot e$ by (9.4). Near p_0, $\Omega_0 = (2\pi i)^2 dt \wedge e$ under E_0 (as $x_i = e^{2\pi i \zeta_i}$), so e generates $\mathfrak{K}$ there by Lemma 9.10(ii). Near p_j, Lemma 9.15(c) together with the computation in the proof of Lemma 9.10(ii) says that the holomorphic $\langle \tilde{g}_j^{\log} \rangle$-invariant 3-forms on J_j' are the $F\, ds \wedge e$ with $F \in s^{k_j}e^{\varphi_j}\mathcal{O}(D_j)$; here e is unchanged under the regluing map h_j of §5, by Lemma 9.16(i) (whose proof is independent of the present proposition), so the comparison of the two descriptions over D_j^* is legitimate, and dividing by $f^*dt_j = m_j s^{m_j - 1} ds$ shows that a unit times $s^{-a_j}e$ generates $\mathfrak{K}$ at p_j, the asserted exponent.

Degree. Let F be the function of Lemma 3.10: F is holomorphic on $\mathfrak{h}_z$, satisfies the automorphy law

$$F(gz) = \jmath_g(z)^{-1}F(z) \qquad (g \in \Delta),$$

is nowhere zero on $\mathfrak{h}_z^\circ$, vanishes to order a_j in the linearising coordinate s_j at the elliptic point z_j ($j = 1, 2$), and has a simple pole at the cusp in the coordinate t_c. Since $\tilde{g}^* e = \det R_g\, e = j_g^{-1} e$ (Lemma 9.10(i) and Lemma 9.14(ii)), the product

$$\mathsf{e} := F^{-1} e$$

is Δ-invariant, i.e. a section of $\mathfrak{K}$ over B°, and it is nowhere vanishing there, both F and e being nowhere zero on $\mathfrak{h}_z^\circ$. Its behaviour at the three remaining points is read off from the frames just determined. At p_j the frame of $\mathfrak{K}$ is $s^{-a_j} e$, and $\mathsf{e} = (s^{a_j} F^{-1}) \cdot (s^{-a_j} e)$ with $s^{a_j} F^{-1}$ a unit at $s = 0$; so e extends across p_j and generates $\mathfrak{K}$ there. At p_0 the frame is e, and F^{-1} is a unit times t_c; so e extends across p_0 with a simple zero. Hence e is a global section of $\mathfrak{K}$ with

$$\operatorname{div}(\mathsf{e}) = p_0, \qquad \text{so} \qquad \mathfrak{K} \cong \mathcal{O}_B(p_0) = \mathcal{O}_B(1), \qquad \deg \mathfrak{K} = 1.$$

The elliptic-surface reading (not used below). The degree just computed agrees with the elliptic-surface reading of the same data, the fibre types being forced by the monodromy together with $e(Y) = 12$. Let $Y \to B$ be the relatively minimal elliptic surface with *this* homological monodromy — of orders 3 and 4 at p_1, p_2 (orders in $SL_2(\mathbb{Z})$, the sign of the lift being fixed by Lemma 9.14) and unipotent at p_0 — so that its j-invariant is $j_Y = 1728\, t$; its Hodge bundle has degree $e(Y)/12$. Here j_Y is holomorphic on $B \smallsetminus \{p_0\}$ and τ is holomorphic on $\mathfrak{h}_z^\circ$ with $\operatorname{Im} \tau > 0$, so the period map does not degenerate over B°: Y has no singular fibres outside $\Sigma = \{p_0, p_1, p_2\}$; the only pole of j_Y is at p_0 and is simple, so the fibre there is of type I_1 or I_1^* ($n = \operatorname{ord}_{p_0}(j_Y)_\infty = 1$); and the fibres over p_1, p_2 are the ones attached to the elliptic points ρ, i with monodromy of orders 3 and 4, hence of type IV or IV^* at p_1 and III or III^* at p_2. The corresponding local Euler numbers are 1 or 7, 4 or 8, and 3 or 9, and $(1, 8, 3)$ is the unique choice with $e(Y) = 12 \deg \mathfrak{K} = 12$; these are precisely the numbers 1 at the cusp and $12 a_j / m_j = 8, 3$ at the two elliptic points.

Canonical bundle. Put $L := K_X \otimes \mathcal{O}(-2S_2)$. By Lemma 9.10 and $K_X|_{F_b} \cong \mathcal{O}$ for $b \in B^\circ$, L restricts trivially to every fibre with its scheme structure; here $h^0(\mathcal{O}_{m_j S_j}) = 1$ because $\mathcal{O}_{m_j S_j}$ is filtered with quotients $\mathcal{O}_{S_j}(-kS_j)$, $0 \le k < m_j$, and $\mathcal{O}_{S_j}(-S_j)$ has order exactly m_j in $\operatorname{Pic}^0(S_j)$ (§5). By Grauert's theorem [GrRe], [BaSt], $M := f_* L$ is invertible and $f^* M \to L$ an isomorphism. Since $f_* \mathcal{O}(2S_2) = \mathcal{O}_B$ ($0 \le 2 < m_2$),

$$\mathfrak{K} = f_*\big(K_X \otimes f^* K_B^{-1}\big) = M(2),$$

so $\deg M = -1$ and $K_X = f^* M \otimes \mathcal{O}(2S_2)$.

Non-torsion. As divisors $4S_2 = f^* p_2$, so $\mathcal{O}(8S_2) \cong f^* \mathcal{O}(2)$ and

$$K_X^{\otimes 4} \cong f^* \mathcal{O}(-4) \otimes f^* \mathcal{O}(2) = f^* \mathcal{O}(-2).$$

If $K_X^{\otimes n} \cong \mathcal{O}_X$ for some $n \ge 1$, then $f^* \mathcal{O}(-2n) \cong K_X^{\otimes 4n} \cong \mathcal{O}_X$, whereas $h^0(X, f^* \mathcal{O}(-2n)) = h^0(B, \mathcal{O}(-2n)) = 0$ by $f_* \mathcal{O}_X = \mathcal{O}_B$ (Proposition 9.13) and $h^0(\mathcal{O}_X) = 1$. Hence K_X is not torsion. $\square$

9.3 Direct images of $\mathcal{O}_X$ and the numbers $h^{0,q}$

We first compute the coherent cohomology of the degenerate fibre W and compare it with constant coefficients. Recall from §4 the normalisation $\nu \colon \mathrm{dP}_6 \to W$ of the (non-simple) normal-crossing surface W: dP_6 is a smooth rational surface containing the hexagon $\partial = C_1 \cup \cdots \cup C_6$ of (-1)-curves in cyclic order, with vertices $p_{i,i+1} := C_i \cap C_{i+1}$ (indices mod 6), ν is injective off ∂, and ν identifies C_i with C_{i+3}; the double locus is $\mathsf{D} = \mathsf{D}_1 \cup \mathsf{D}_2 \cup \mathsf{D}_3$ with

$$\mathsf{D}_k := \nu(C_k) = \nu(C_{k+3}) \cong \mathbb{P}^1 \qquad (k = 1, 2, 3),$$

and W has exactly two triple points P, Q. Put $\iota_k \colon \mathsf{D}_k \hookrightarrow W$ and

$$\nu_k^+ := \nu|_{C_k} \colon C_k \xrightarrow{\sim} \mathsf{D}_k, \qquad \nu_k^- := \nu|_{C_{k+3}} \colon C_{k+3} \xrightarrow{\sim} \mathsf{D}_k.$$

We also recall from §4 the combinatorial labelling used below: $\rho_1, \ldots, \rho_6$ are the six lattice points adjacent to the origin in the A_2-triangulation $\mathcal{T}_{A_2}$, taken in cyclic order,

$$\rho_1 = e_1, \quad \rho_2 = e_1 - e_2, \quad \rho_3 = -e_2, \quad \rho_4 = -e_1, \quad \rho_5 = e_2 - e_1, \quad \rho_6 = e_2$$

(indices mod 6, so that $\rho_{i+3} = -\rho_i$ and $\rho_{i+1} - \rho_i = \rho_{i+2}$); $C_i \subset \mathrm{dP}_6$ is the toric curve attached to the cone over the edge $[0, \rho_i]$, and $O_{\{0,\rho_i,\rho_{i+1}\}}$ denotes the torus-fixed point attached to the cone over the triangle $\{0, \rho_i, \rho_{i+1}\}$, so that $p_{i,i+1} = C_i \cap C_{i+1} = O_{\{0,\rho_i,\rho_{i+1}\}}$.

Lemma 9.12. (i) Define $\mathbb{C}$-linear sheaf maps on W by $\nu^\sharp(f) = f \circ \nu$,

$$\delta_0(f)_k := \big(f|_{C_k}\big) \circ (\nu_k^+)^{-1} - \big(f|_{C_{k+3}}\big) \circ (\nu_k^-)^{-1} \quad \text{on } \mathsf{D}_k \ (k = 1, 2, 3),$$
$$\delta_1(g_1, g_2, g_3) := \big(g_1(P) - g_2(P) + g_3(P),\ g_1(Q) - g_2(Q) + g_3(Q)\big)$$

(over an open $U \subseteq W$ the P-component is present only if $P \in U$, and likewise for Q). Then

$$0 \longrightarrow \mathcal{O}_W \xrightarrow{\nu^\sharp} \nu_*\mathcal{O}_{\mathrm{dP}_6} \xrightarrow{\delta_0} \bigoplus_{k=1}^{3} \iota_{k*}\mathcal{O}_{\mathsf{D}_k} \xrightarrow{\delta_1} \mathbb{C}_P \oplus \mathbb{C}_Q \longrightarrow 0 \tag{9.5}$$

is an exact sequence of sheaves on W, and the same formulas define an exact sequence

$$0 \longrightarrow \mathbb{C}_W \longrightarrow \nu_*\mathbb{C}_{\mathrm{dP}_6} \xrightarrow{\delta_0} \bigoplus_{k=1}^{3} \iota_{k*}\mathbb{C}_{\mathsf{D}_k} \xrightarrow{\delta_1} \mathbb{C}_P \oplus \mathbb{C}_Q \longrightarrow 0, \tag{9.6}$$

the inclusions of constants giving a termwise map (9.6)$\to$(9.5).

(ii) $h^q(W, \mathcal{O}_W) = 1, 2, 1$ for $q = 0, 1, 2$; more precisely $H^q(W, \mathcal{O}_W)$ is the q-th cohomology of the simplicial cochain complex $\mathbb{C} \to \mathbb{C}^3 \to \mathbb{C}^2$ of the dual complex of W (one vertex, three edges $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3$, two triangles P, Q), which is the standard two-triangle CW structure of a real 2-torus.

(iii) The map $H^1(W; \mathbb{C}) \to H^1(W, \mathcal{O}_W)$ induced by $\mathbb{C}_W \subset \mathcal{O}_W$ is an isomorphism, and $H^2(W; \mathbb{C}) \to H^2(W, \mathcal{O}_W)$ is surjective and injective on $\ker\big(\nu^*\colon H^2(W; \mathbb{C}) \to H^2(\mathrm{dP}_6; \mathbb{C})\big)$; that is, a class $\alpha \in H^2(W; \mathbb{C})$ with $\nu^*\alpha = 0$ has zero image in $H^2(W, \mathcal{O}_W)$ if and only if $\alpha = 0$.

(iv) The cup product $\Lambda^2 H^1(W, \mathcal{O}_W) \to H^2(W, \mathcal{O}_W)$ is an isomorphism.

Proof. Combinatorics of ν. By §4 the gluing $C_i \to C_{i+3}$ is induced by the shear $\Psi_{\bar\lambda}$ with $B_0\bar\lambda = -\rho_i$, which carries the vertex $p_{i,i+1} = O_{\{0,\rho_i,\rho_{i+1}\}}$ to $O_{\{-\rho_i,0,\rho_{i+1}-\rho_i\}} = O_{\{0,\rho_{i+3},\rho_{i+2}\}} = p_{i+2,i+3}$, using $\rho_{i+1} - \rho_i = \rho_{i+2}$ and $-\rho_i = \rho_{i+3}$. Hence the six vertices fall into two classes according to parity,

$$P = \nu(p_{12}) = \nu(p_{34}) = \nu(p_{56}), \qquad Q = \nu(p_{23}) = \nu(p_{45}) = \nu(p_{61}),$$

and these are the two triple points. The three branches of W at P are the germs b_1, b_2, b_3 of dP_6 at p_{12}, p_{34}, p_{56}, with $b_1 \supset C_1, C_2$, $b_2 \supset C_3, C_4$, $b_3 \supset C_5, C_6$; consequently

$$\mathsf{D}_1 = b_1 \cap b_2 \ (C_1 \subset b_1,\ C_4 \subset b_2), \quad \mathsf{D}_2 = b_1 \cap b_3 \ (C_2 \subset b_1,\ C_5 \subset b_3), \quad \mathsf{D}_3 = b_2 \cap b_3 \ (C_3 \subset b_2,\ C_6 \subset b_3),$$

so that each pair of branches meets along exactly one D_k and the local model of W at P is $\{xyz = 0\} \subset \mathbb{C}^3$ (as it must be, since $t = z_0z_1z_2$ at the corresponding toric point and the Γ-quotient is a local isomorphism). At Q the branches are c_1, c_2, c_3, the germs at p_{23}, p_{45}, p_{61}, with $c_1 \supset C_2, C_3$, $c_2 \supset C_4, C_5$, $c_3 \supset C_6, C_1$, so that

$$\mathsf{D}_1 = c_3 \cap c_2, \qquad \mathsf{D}_2 = c_1 \cap c_2, \qquad \mathsf{D}_3 = c_1 \cap c_3.$$

At a point of $\mathsf{D}_k \smallsetminus \{P, Q\}$ the model is $\{xy = 0\} \times \mathbb{C}$, the two branches carrying C_k and C_{k+3}; elsewhere W is smooth.

(i) $\delta_1\delta_0 = 0$. Let f be a section of $\nu_*\mathcal{O}_{\mathrm{dP}_6}$ near P and let f_1, f_2, f_3 be its values at p_{12}, p_{34}, p_{56}, i.e. on the branches b_1, b_2, b_3. By the table above, $\delta_0(f)_1(P) = f_1 - f_2$ (the $+$ from $C_1 \subset b_1$, the $-$ from $C_4 \subset b_2$), $\delta_0(f)_2(P) = f_1 - f_3$ and $\delta_0(f)_3(P) = f_2 - f_3$, so

$$\delta_1\delta_0(f)(P) = (f_1 - f_2) - (f_1 - f_3) + (f_2 - f_3) = 0.$$

At Q, with f_i' the value on c_i, one gets $\delta_0(f)_1(Q) = f_3' - f_2'$, $\delta_0(f)_2(Q) = f_1' - f_2'$, $\delta_0(f)_3(Q) = f_1' - f_3'$, and $(f_3' - f_2') - (f_1' - f_2') + (f_1' - f_3') = 0$. Thus the single global sign convention ("$+$ on C_k, $-$ on C_{k+3}", and $\delta_1 = g_1 - g_2 + g_3$ at *both* triple points) makes (9.5) a complex; it is the Čech differential of the branch covering in the orderings (b_1, b_2, b_3) at P and (c_1, c_3, c_2) at Q (the sign pattern of δ_0 at Q being the tournament $c_1 > c_3 > c_2$).

(i) Exactness. This is local, and by the description above there are three kinds of points. (Equivalently, one may quote the gluing Lemma of §10, which presents $\mathcal{O}_W$ as the fibre product of $\nu_*\mathcal{O}_{\mathrm{dP}_6}$ over the $\mathcal{O}_{\mathsf{D}_k}$; the computations below are its local form, and we give them in full.)

(a) W smooth at x: ν is biholomorphic near x and the remaining terms of (9.5) vanish near x, so the sequence reduces to $0 \to \mathcal{O} \xrightarrow{=} \mathcal{O} \to 0$.

(b) $x \in \mathsf{D}_k \smallsetminus \{P, Q\}$: the model is $A := \mathbb{C}\{x, y, z\}/(xy)$ with branches $A_1 = \mathbb{C}\{y, z\}$ $(x = 0)$ and

$A_2 = \mathbb{C}\{x,z\}$ $(y=0)$ and double curve $A_{12} = \mathbb{C}\{z\}$, and (9.5) becomes

$$0 \to A \to A_1 \oplus A_2 \xrightarrow{(a,b)\mapsto a|_{y=0} - b|_{x=0}} A_{12} \to 0.$$

Injectivity holds because $(x) \cap (y) = (xy)$ in $\mathbb{C}\{x,y,z\}$. If $a(0,z) = b(0,z)$, then $f := a(y,z) + b(x,z) - a(0,z)$ lies in $\mathbb{C}\{x,y,z\}$ and restricts to a on $\{x=0\}$ and to b on $\{y=0\}$, so the image is the whole kernel; and $(g(z),0) \mapsto g$ gives surjectivity.

(c) $x = P$ (or Q): the model is $A = \mathbb{C}\{x,y,z\}/(xyz)$ with branches $A_1 = \mathbb{C}\{y,z\}$, $A_2 = \mathbb{C}\{x,z\}$, $A_3 = \mathbb{C}\{x,y\}$, double curves $A_{12} = \mathbb{C}\{z\}$, $A_{13} = \mathbb{C}\{y\}$, $A_{23} = \mathbb{C}\{x\}$, triple point $A_{123} = \mathbb{C}$, and by the combinatorial table (9.5) becomes (after the relabelling of branches recorded above)

$$0 \to A \to \bigoplus_i A_i \xrightarrow{d_0} \bigoplus_{i<j} A_{ij} \xrightarrow{d_1} \mathbb{C} \to 0, \qquad d_0(f)_{ij} = f_i| - f_j|, \quad d_1(g) = g_{12}(0) - g_{13}(0) + g_{23}(0).$$

Injectivity: $(x) \cap (y) \cap (z) = (xyz)$. Exactness at $\bigoplus A_i$: suppose $f_i = f_j$ on each line L_{ij} and put

$$f := f_3(x,y) + \big[f_1(y,z) - f_1(y,0)\big] + \big[f_2(x,z) - f_2(x,0)\big] - \big[f_2(0,z) - f_2(0,0)\big].$$

Then $f|_{z=0} = f_3$; $f|_{x=0} = f_3(0,y) + f_1(y,z) - f_1(y,0) = f_1(y,z)$ because $f_3(0,y) = f_1(y,0)$ on L_{13}; and $f|_{y=0} = f_3(x,0) + \big[f_1(0,z) - f_1(0,0)\big] + f_2(x,z) - f_2(x,0) - \big[f_2(0,z) - f_2(0,0)\big] = f_2(x,z)$, using $f_3(x,0) = f_2(x,0)$ on L_{23} and $f_1(0,z) = f_2(0,z)$ on L_{12}. Exactness at $\bigoplus A_{ij}$: $d_1 d_0 = 0$ was checked above; conversely, if $g_{12}(0) - g_{13}(0) + g_{23}(0) = 0$, put $f_3 := 0$, $f_1(y,z) := g_{13}(y) + g_{12}(z) - g_{12}(0)$, $f_2(x,z) := g_{23}(x)$. Then $f_1| - f_3| = g_{13}$ on L_{13}, $f_2| - f_3| = g_{23}$ on L_{23}, and on L_{12}, $f_1| - f_2| = g_{13}(0) + g_{12}(z) - g_{12}(0) - g_{23}(0) = g_{12}(z)$ by the relation. Finally $(c,0,0) \mapsto c$ gives surjectivity of d_1.

In each of (a), (b), (c) the same computation with $\mathcal{O}$ replaced by germs of locally constant functions is the statement that $0 \to \mathbb{C} \to \mathbb{C} \to 0$, $0 \to \mathbb{C} \to \mathbb{C}^2 \to \mathbb{C} \to 0$, $0 \to \mathbb{C} \to \mathbb{C}^3 \to \mathbb{C}^3 \to \mathbb{C} \to 0$ are exact, i.e. the reduced simplicial cochain complexes of a point, a 1-simplex and a 2-simplex; so (9.6) is exact as well, and the termwise inclusions of constants commute with $\nu^\sharp, \delta_0, \delta_1$ by construction.

(ii). By (i), $\mathcal{O}_W$ is quasi-isomorphic to the complex $K^\bullet := \big[\nu_*\mathcal{O}_{\mathrm{dP}_6} \to \bigoplus_k \iota_{k*}\mathcal{O}_{\mathrm{D}_k} \to \mathbb{C}_P \oplus \mathbb{C}_Q\big]$ placed in degrees $0,1,2$. The maps ν and ι_k are finite, so $H^q(W, \nu_*\mathcal{O}_{\mathrm{dP}_6}) = H^q(\mathrm{dP}_6, \mathcal{O}_{\mathrm{dP}_6}) = (\mathbb{C},0,0)$ (dP_6 is a smooth rational surface) and $H^q(W, \iota_{k*}\mathcal{O}_{\mathrm{D}_k}) = H^q(\mathbb{P}^1, \mathcal{O}) = (\mathbb{C},0)$. Hence in the hypercohomology spectral sequence $E_1^{p,q} = H^q(W, K^p) \Rightarrow H^{p+q}(W, \mathcal{O}_W)$ all rows $q \geq 1$ vanish and

$$H^n(W, \mathcal{O}_W) = H^n\big(\Gamma(W, K^\bullet)\big), \qquad \Gamma(W, K^\bullet)\colon\ \mathbb{C} \xrightarrow{d_0 = 0} \mathbb{C}^3 \xrightarrow{d_1} \mathbb{C}^2, \quad d_1(a_1,a_2,a_3) = (a_1 - a_2 + a_3)(1,1);$$

here $d_0 = 0$ because a constant function on dP_6 has equal restrictions to C_k and C_{k+3}. This complex is precisely the simplicial cochain complex of the dual complex described in the statement, and

$$h^0(\mathcal{O}_W) = 1, \qquad h^1(\mathcal{O}_W) = \dim\ker d_1 = 2, \qquad h^2(\mathcal{O}_W) = \dim\operatorname{coker} d_1 = 1,$$

so $\chi(\mathcal{O}_W) = 0$, the same numbers as for a smooth fibre.

(iii). Run the same spectral sequence for (9.6): its row $q = 0$ is the *same* complex $\Gamma(W, K^\bullet)$ (all the pieces dP_6, D_k, $\{P\}, \{Q\}$ being compact and connected), its row $q = 1$ vanishes ($H^1(\mathrm{dP}_6;\mathbb{C}) = H^1(\mathbb{P}^1;\mathbb{C}) = 0$), and its row $q = 2$ is $H^2(\mathrm{dP}_6;\mathbb{C}) = \mathbb{C}^4 \to \bigoplus_k H^2(\mathrm{D}_k;\mathbb{C}) = \mathbb{C}^3 \to 0$. The comparison map of spectral sequences induced by $\mathbb{C} \to \mathcal{O}$ is the identity on row 0 and zero on the rows $q \geq 1$ (whose targets vanish).

In total degree 1 the only surviving term on either side is $E_2^{1,0} = \ker d_1$, and the comparison is the identity there; hence $H^1(W;\mathbb{C}) \to H^1(W, \mathcal{O}_W)$ is an isomorphism (both spaces being $\ker d_1 \cong \mathbb{C}^2$). In total degree 2, $H^2(W, \mathcal{O}_W) = E_2^{2,0} = \operatorname{coker} d_1$, while for $\mathbb{C}$-coefficients $E_\infty^{2,0} = \operatorname{coker} d_1$ as well (the only possibly incoming differential is d_2 from $E_2^{0,1} = 0$), $E_\infty^{1,1} = 0$, and $E_\infty^{0,2} \subseteq H^2(\mathrm{dP}_6;\mathbb{C})$ with edge map the restriction ν^*. Writing $F^\bullet$ for the filtration of $H^2(W;\mathbb{C})$, we therefore have

$$\ker\nu^* = F^1H^2(W;\mathbb{C}) = F^2H^2(W;\mathbb{C}) = E_\infty^{2,0} = \operatorname{coker} d_1,$$

and the comparison map carries this subspace isomorphically onto $E_2^{2,0} = H^2(W, \mathcal{O}_W)$. In particular $H^2(W;\mathbb{C}) \to H^2(W, \mathcal{O}_W)$ is surjective and injective on $\ker\nu^*$, which is (iii).

(iv). The sheaf inclusion $\mathbb{C}_W \subset \mathcal{O}_W$ induces a ring homomorphism $H^\bullet(W;\mathbb{C}) \to H^\bullet(W, \mathcal{O}_W)$, by naturality of the cup product. Let $\gamma, u \in H^1(W;\mathbb{Z})$ be the basis of Proposition 7.12 (so that $\mathrm{sp}^1(\gamma,u) = (\gamma,u) \in V^{T_0}$), and let $\gamma^{0,1}, u^{0,1}$ be their images in $H^1(W, \mathcal{O}_W)$; by (iii) these form a basis of $H^1(W, \mathcal{O}_W)$. By Proposition 7.12, $\gamma \cup u = \mathbb{T}_B^\vee \neq 0$, the class dual to $\mathbb{T}_B$ in the basis $([\mathrm{D}_1], [\mathrm{D}_2], [\mathrm{D}_3], [\mathbb{T}_B])$ of $H_2(W)$. Now

$H_2(\mathrm{dP}_6;\mathbb{Z}) \cong \mathbb{Z}^4$ is generated by the classes $[C_1], \ldots, [C_6]$ of the six hexagon (-1)-curves alone (they span, subject to two relations), and $\nu_*[C_i] = [\mathsf{D}_{i \bmod 3}]$, on which $\mathbb{T}_B^\vee$ vanishes; hence, by the projection formula, $\nu^*(\mathbb{T}_B^\vee) = 0$. By (iii), a non-zero class in $\ker \nu^*$ has non-zero image in $H^2(W, \mathcal{O}_W)$; hence

$$\gamma^{0,1} \cup u^{0,1} = \text{image of } \gamma \cup u \neq 0 \quad \text{in } H^2(W, \mathcal{O}_W).$$

As $h^2(\mathcal{O}_W) = 1$ and $\Lambda^2 H^1(W, \mathcal{O}_W)$ is spanned by $\gamma^{0,1} \wedge u^{0,1}$, the cup product $\Lambda^2 H^1(W, \mathcal{O}_W) \to H^2(W, \mathcal{O}_W)$ is an isomorphism. □

Proposition 9.13. *$f_*\mathcal{O}_X = \mathcal{O}_B$, $R^1 f_*\mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-1)$, $R^2 f_*\mathcal{O}_X \cong \mathcal{O}(-1)$. Consequently $h^{0,0} = h^{0,1} = 1$, $h^{0,2} = h^{0,3} = 0$, $h^{3,3} = h^{3,2} = 1$, $h^{3,1} = h^{3,0} = 0$.*

Proof. f is proper with connected fibres onto the normal curve B, so $f_*\mathcal{O}_X = \mathcal{O}_B$.

Step 0: flatness, and the topological input. The map f is flat. Indeed, X is a connected complex manifold, B a smooth curve and f non-constant, hence open; equivalently, for each $x \in X$ no non-zero germ in $\mathcal{O}_{B,f(x)}$ vanishes identically near x, so $\mathcal{O}_{X,x}$ is torsion free — and therefore flat — over the discrete valuation ring $\mathcal{O}_{B,f(x)}$. The same applies to the restrictions $f_0 \colon N_0 \to D_0$ and $f|_{N_j} \colon N_j \to D_j$. Moreover the scheme-theoretic fibre of f_0 over p_0 is the *reduced* divisor W: all rays of $\mathfrak{F}$ lie at height 1, so $t^{-1}(0) = \sum_v D_v$ is reduced (Lemma 9.10(ii)), hence so is its quotient W.

The following topological fact will be used:

(T) The inclusion $W \hookrightarrow N_0$ induces an isomorphism on fundamental groups, both equal to $\bar{\Lambda}$: $N_0 = Y_\varepsilon/\Gamma$ with $\pi_1(Y_\varepsilon) = 1$ (Corollary 4.8), and $W = t^{-1}(0)/\Gamma$ with $t^{-1}(0)$ simply connected, being a deformation retract of a tube $\{|t| < \rho\} \subset Y$ (Lemma A.2(6), a statement upstairs) which is simply connected by the argument of Corollary 4.8; Γ acts freely on both and the inclusion is Γ-equivariant. Hence restriction $H^1(N_0;\mathbb{C}) \to H^1(W;\mathbb{C})$ is an isomorphism, both being $\mathrm{Hom}(\bar{\Lambda}, \mathbb{C})$. Moreover, by Proposition 7.12, for $b \in D_0^*$ the restriction

$$H^1(W;\mathbb{C}) \xrightarrow{\mathrm{sp}^1} H^1(F_b;\mathbb{C}) = V \otimes \mathbb{C} \tag{9.7}$$

is injective with image $V^{T_0} \otimes \mathbb{C} = \mathbb{C}\gamma \oplus \mathbb{C}u$, carrying the basis γ, u of $H^1(W;\mathbb{C})$ of Lemma 9.12(iv) to $\gamma, u \in V$.

Step 1: local freeness and base change. Over B° the fibres are complex 2-tori with $h^q(\mathcal{O}_{F_b}) = (1,2,1)$. Near p_0 the fibre of the proper flat map f_0 over p_0 is the reduced surface W, again with $h^q(\mathcal{O}_W) = (1,2,1)$ by Lemma 9.12(ii). Hence $b \mapsto h^q(f^{-1}(b), \mathcal{O})$ is constant on $B^\circ \cup \{p_0\}$ for every q, and Grauert's theorem [GrRe], [BaSt, III.§3] applies there: $R^q f_*\mathcal{O}_X$ is locally free of rank $1,2,1$ on $B^\circ \cup \{p_0\}$ and *every* base-change map

$$R^q f_*\mathcal{O}_X \otimes k(b) \longrightarrow H^q(f^{-1}(b), \mathcal{O}), \qquad q = 0,1,2,$$

is an isomorphism, compatibly with cup products. Near p_j the scheme fibre is the non-reduced $m_j S_j$ and we argue instead on the cover: $N_j = J_j'/\langle \tilde{g}_j^{\log} \rangle$ with $\tilde{f} \colon J_j' \to \Delta_j$ smooth and proper, and taking $\mathbb{Z}/m_j$-invariants is exact, so with $\pi(s) = s^{m_j}$,

$$R^q f_*\mathcal{O}_{N_j} = \left(\pi_* R^q \tilde{f}_* \mathcal{O}_{J_j'}\right)^{\mathbb{Z}/m_j};$$

decomposing $R^q \tilde{f}_*\mathcal{O}$ into eigenframes and using $\pi_*\mathcal{O}_{\Delta_j} = \bigoplus_{r<m_j} s^r \mathcal{O}_{D_j}$, these invariants are free of ranks $1,2,1$. Thus all $R^q f_*\mathcal{O}_X$ are locally free on B, of ranks $1,2,1$; write $G := R^1 f_*\mathcal{O}_X$.

Step 2: $R^2 f_*\mathcal{O}_X$ and $\deg G$. Relative duality [RRV] gives $R^2 f_*\mathcal{O}_X \cong \mathfrak{K}^{-1} \cong \mathcal{O}(-1)$ by Proposition 9.11. The hypotheses are met: f is flat (Step 0) and proper, its fibres — including the non-reduced ones $m_j S_j$ — are Gorenstein, since X is smooth and f flat, so that $\omega_{X/B} = K_X \otimes f^*K_B^{-1}$ is invertible and is a relative dualising sheaf, and the sheaves $R^q f_*\mathcal{O}_X$ are locally free by Step 1; hence relative duality applies. We claim that the relative cup product $\Lambda^2 G \to R^2 f_*\mathcal{O}_X$ is an isomorphism; as both sides are line bundles it suffices to check non-vanishing at each point of B.

Over B° this is, by the base change of Step 1, the isomorphism $\Lambda^2 H^1(\mathcal{O}_{F_b}) \cong H^2(\mathcal{O}_{F_b})$ on a complex 2-torus. At p_0 it is, again by base change, the map $\Lambda^2 H^1(W, \mathcal{O}_W) \to H^2(W, \mathcal{O}_W)$, an isomorphism by Lemma 9.12(iv).

At p_j we work on the cover. Linearise the action of the cyclic group $\mathbb{Z}/m_j$ on the free $\mathcal{O}(\Delta_j)$-module $R^1\tilde{f}_*\mathcal{O}_{J'_j}$ of rank 2 (average over the group) and choose a frame v_1, v_2 on which the generator acts by constants ν_1, ν_2; then it acts on the frame $v_1 \wedge v_2$ of Λ^2 by $\nu_1\nu_2$. (Such an eigenframe exists, and may be chosen so as to contain a prescribed invariant nowhere-vanishing section v_1: after averaging, the generator acts semilinearly over $s \mapsto \zeta_j s$ and induces on the fibre at $s = 0$ a linear map of order dividing m_j, hence diagonalisable; diagonalise it with first eigenvector the value $v_1(0) \neq 0$, lift the remaining eigenvector arbitrarily and average it against its eigenvalue — the resulting eigensections still form a frame by Nakayama.) A section $c(s)v_i$ is invariant iff $c(s) = \nu_i c(\zeta_j s)$, i.e. iff $c \in s^{r_i}\mathcal{O}(D_j)$ with the unique $r_i \in \{0, \dots, m_j - 1\}$ such that $\nu_i \zeta_j^{r_i} = 1$; likewise the invariants of Λ^2 are generated by $s^r(v_1 \wedge v_2)$ with $r \in \{0, \dots, m_j - 1\}$, $r \equiv r_1 + r_2 \pmod{m_j}$. Now:

- $r_1 = 0$ for a suitable v_1: by Step 3 below, $\gamma^{0,1}$ is an invariant nowhere-zero section of $R^1\tilde{f}_*\mathcal{O}$ over Δ_j, which we take as v_1 (so $\nu_1 = 1$; this is legitimate by the parenthetical remark above);
- $r = a_j$: indeed $\Lambda^2 R^1\tilde{f}_*\mathcal{O} \cong R^2\tilde{f}_*\mathcal{O} \cong (\tilde{f}_*\omega_{J'_j/\Delta_j})^{-1}$, and $\tilde{f}_*\omega_{J'_j/\Delta_j}$ has the frame e on which the generator acts by $\det R_{g_j}$, of value $\zeta_j^{a_j}$ at $s = 0$ (Lemma 9.10(i)); hence the generator acts on the dual frame by $\zeta_j^{-a_j}$ and r is determined by $\zeta_j^{-a_j}\zeta_j^r = 1$, i.e. $r = a_j$, which lies in $\{0, \dots, m_j - 1\}$ because $0 < a_j < m_j$ (Lemma 9.10(i)).

Therefore $r_2 \equiv r_1 + r_2 = r = a_j \pmod{m_j}$ with $r_2, a_j \in \{0, \dots, m_j - 1\}$, so

$$r_1 + r_2 = a_j < m_j, \qquad r_2 = a_j = r,$$

and $v_1 \wedge (s^{r_2}v_2) = s^r(v_1 \wedge v_2)$: the cup product carries the invariant frame of $\Lambda^2\big(R^1 f_*\mathcal{O}_{N_j}\big)$ to the invariant frame of $R^2 f_*\mathcal{O}_{N_j}$, i.e. it is an isomorphism at p_j (no "carrying" mod m_j occurs, precisely because $r_1 + r_2 < m_j$).

Hence $\Lambda^2 G \cong R^2 f_*\mathcal{O}_X \cong \mathcal{O}(-1)$ and $\deg G = -1$.

Step 3: $\gamma^{0,1}$ away from p_0. The class $\gamma \in V$ is T_1- and T_2-invariant, hence a flat section of $R^1 f_*\mathbb{Z}$ over B°, and its image $\gamma^{0,1}$ in $G|_{B^\circ} = (V \otimes \mathcal{O})/F^1$ is holomorphic and nowhere zero there, since $a\sigma_1 + b\sigma_2 = \gamma$ forces $a = b = 0$ (compare the w- and δ-coefficients). At p_1, p_2 the local system pulls back to the *constant* system on the disc Δ_j (the monodromy of g_j having been unwound there), γ is a constant, hence invariant, section of it, and the same comparison of coefficients at $z = z_j$ gives $\gamma^{0,1}(z_j) \neq 0$; as $J'_j \to N_j$ is étale, $\gamma^{0,1}$ is an invariant nowhere-zero section of $R^1\tilde{f}_*\mathcal{O}_{J'_j}$ over Δ_j and descends to a nowhere-zero section of G over D_j.

Step 4: $\gamma^{0,1}$ and $u^{0,1}$ at p_0. The inclusion $\mathbb{C} \subset \mathcal{O}_{N_0}$ induces an $\mathcal{O}$-linear map

$$R^1 f_{0*}\mathbb{C}_{N_0} \otimes \mathcal{O}_{D_0} \longrightarrow R^1 f_{0*}\mathcal{O}_{N_0} = G|_{D_0},$$

defined on all of D_0, and restriction of cohomology classes to the fibres gives $H^1(N_0;\mathbb{C}) \to H^0(D_0, R^1 f_{0*}\mathbb{C})$. By (T) the restriction $H^1(N_0;\mathbb{C}) \to H^1(W;\mathbb{C})$ is an isomorphism, so the basis γ, u of $H^1(W;\mathbb{C})$ of Lemma 9.12(iv) lifts to classes $c_\gamma, c_u \in H^1(N_0;\mathbb{C})$ and produces holomorphic sections

$$\gamma^{0,1},\ u^{0,1} \in H^0(D_0, G)$$

over the *whole* disc. Their values at p_0 are computed by the base-change isomorphism $G \otimes k(p_0) \cong H^1(W, \mathcal{O}_W)$ of Step 1 together with the functoriality of $\mathbb{C} \subset \mathcal{O}$ for $W \subset N_0$: they are the images of $c_\gamma|_W = \gamma$ and $c_u|_W = u$ under

$$H^1(W;\mathbb{C}) \longrightarrow H^1(W, \mathcal{O}_W),$$

which is an isomorphism by Lemma 9.12(iii). Hence $\gamma^{0,1}(p_0), u^{0,1}(p_0)$ form a basis of $G \otimes k(p_0)$; in particular $\gamma^{0,1}(p_0) \neq 0$, so $\mathcal{O}\gamma^{0,1} \subset G$ is a subbundle near p_0, and $(\gamma^{0,1}, u^{0,1})$ is a holomorphic frame of G on a neighbourhood of p_0. Finally, over D_0^* the section $\gamma^{0,1}$ just constructed coincides with the one of Step 3: by (9.7) the restriction of c_γ to F_b is $\gamma \in V$, whose image in $(V \otimes \mathcal{O})/F^1$ is the section of Step 3. So the two constructions glue to one global section $\gamma^{0,1}$ of G over B, nowhere vanishing.

Step 5: conclusion. By Steps 3 and 4, $\gamma^{0,1}$ is a nowhere-vanishing global section of G, so $\mathcal{O}\gamma^{0,1} \subset G$ is a subbundle (over B° by the chart condition of §3, at p_j via the covers, at p_0 by Step 4), with quotient the line bundle $\Lambda^2 G \otimes (\mathcal{O}\gamma^{0,1})^{-1} \cong \Lambda^2 G \cong \mathcal{O}(-1)$ by Step 2. As $\mathrm{Ext}^1(\mathcal{O}(-1), \mathcal{O}) = H^1(\mathbb{P}^1, \mathcal{O}(1)) = 0$, the extension splits: $G \cong \mathcal{O} \oplus \mathcal{O}(-1)$.

Since $\dim B = 1$, Leray gives $0 \to H^1(R^{q-1} f_* \mathcal{O}_X) \to H^q(\mathcal{O}_X) \to H^0(R^q f_* \mathcal{O}_X) \to 0$, whence $h^{0,0} = 1$, $h^{0,1} = h^0(G) = 1$, $h^{0,2} = h^1(G) + h^0(\mathcal{O}(-1)) = 0$, $h^{0,3} = h^1(\mathcal{O}(-1)) = 0$; Serre duality gives the $h^{3,q}$. $\square$

9.4 Chern numbers and Riemann–Roch

By §7 we have $H^2(X;\mathbb{Z}) = 0$ and $H^4(X;\mathbb{Z}) = 0$, so $c_1(X) = c_2(X) = 0$ integrally, and every Chern number involving c_1 or c_2 vanishes; Gauss–Bonnet and §7 give

$$c_3(X) = e(X) = e(W) + e(3S_1) + e(4S_2) = 2 + 0 + 0.$$

Hirzebruch–Riemann–Roch [Hir], [AS] gives $\chi(\mathcal{O}_X) = \frac{1}{24} c_1 c_2 = 0$, matching $1 - 1 + 0 - 0 = 0$ from Proposition 9.13; for T_X one has $\mathrm{ch}(T_X) = 3 + \frac{1}{2} c_3$, $\mathrm{td}(X) = 1$, so $\chi(T_X) = \frac{1}{2} c_3 = 1$. By Serre duality $\chi(\Omega^3_X) = -\chi(\mathcal{O}_X) = 0$ and $\chi(\Omega^2_X) = -\chi(\Omega^1_X)$, so $2 = c_3 = \sum_p (-1)^p \chi(\Omega^p_X) = -2\chi(\Omega^1_X)$, i.e. $\chi(\Omega^1_X) = -1$. This proves Theorem 9.1(iv).

9.5 Holomorphic forms on X

In this subsection we prove Theorem 9.1(vi):

$$H^0(X, \Omega^p_X) = 0 \qquad (p = 1, 2, 3).$$

The method is elementary and explicit: a holomorphic form on X is pulled back to the three uniformisations of §6 — the cover $\mathfrak{h}_z \times \mathbb{C}^2$ of J, the étale covers $J'_j \to N_j$ of §5, and the toric cover $Y_\varepsilon \to N_0$ of §4 — where its coefficients become functions of z alone, subject to an automorphy law under Δ and to vanishing conditions at the three points of Σ; the resulting objects are sections of negative line bundles on $\mathbb{P}^1$.

Convention. Throughout §9.5 a dot denotes d/dz on $\mathfrak{h}_z$ (so $\dot\tau = d\tau/dz$, $\dot\Pi = d\Pi/dz$, $\dot g(z)$ the derivative of the Möbius transformation $g \in \Delta$); primes are reserved for derivatives in the base coordinates t_c, t_j. We write $\zeta = (\zeta_1, \zeta_2)$ for the fibre coordinates on $\mathfrak{h}_z \times \mathbb{C}^2$, $c = (c_1, c_2)$ and $b = (b_1, b_2)$ for *row* vectors, so that $c \cdot d\zeta = c_1 d\zeta_1 + c_2 d\zeta_2$, and

$$T_\lambda(z, \zeta) := (z, \zeta + \Pi(z)\lambda) \ \ (\lambda \in \Lambda), \qquad \tilde g(z, \zeta) := (gz, R_g(z)\zeta) \ \ (g \in \Delta),$$

so that $\mathcal{T}|_U = (U \times \mathbb{C}^2)/\Lambda$ for $U \subseteq \mathfrak{h}_z$ open, and $J = (\mathcal{T}|_{\mathfrak{h}_z^\circ})/\Delta$.

Step 0: the automorphy factor

Lemma 9.14. (i) For every $g \in \Delta$ the matrix $R_g(z)$ is lower triangular with second column e_2:

$$R_g(z) = \begin{pmatrix} r_g(z) & 0 \\ * & 1 \end{pmatrix}, \qquad r_g = \det R_g,$$

and $r_{g_1} = -1/\tau$, $r_{g_2} = 1/\tau$, $r_{g_0} = 1$.

(ii) Put $\jmath_g := r_g^{-1}$, so $\jmath_{g_1} = -\tau$, $\jmath_{g_2} = \tau$, $\jmath_{g_0} = 1$. Then $\jmath$ is a cocycle, $\jmath_{gh}(z) = \jmath_g(hz)\,\jmath_h(z)$, and

$$\jmath_g(z) = c_g \tau(z) + d_g, \qquad g \longmapsto \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}, \quad g_1 \mapsto \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the lift with $\jmath_{g_2} = +\tau$ of the homomorphism $\Delta \to PSL_2(\mathbb{Z})$ of §3. Consequently, for the holomorphic 1-form $d\tau = \dot\tau\, dz$ on $\mathfrak{h}_z$,

$$g^*(d\tau) = \jmath_g^{-2}\, d\tau \qquad (g \in \Delta).$$

(The letter j is reserved for the modular invariant; the automorphy cocycle is $\jmath$.)

(iii) $\dot\tau(z) \neq 0$ for $z \in \mathfrak{h}_z^\circ$; hence $\dot Z(z) = \begin{pmatrix} 6\dot\mu & \dot\tau \\ \dot\beta & \dot\mu \end{pmatrix} \neq 0$ for $z \in \mathfrak{h}_z^\circ$, and $\{\dot\tau = 0\}$ is contained in the discrete set $\mathfrak{h}_z \smallsetminus \mathfrak{h}_z^\circ$ of elliptic points.

Proof. (i) By (9.4) the second column of $R_g(z)$ is $\Pi(gz) A_g \hat\delta = \Pi(gz)\hat\delta = e_2$, because $\hat\delta$ spans $\Lambda^{\langle A_1, A_2 \rangle}$ and $\Pi = [Z \mid I]$. Hence R_g is lower triangular with $(2,2)$-entry 1, and $\det R_g = (R_g)_{11} =: r_g$. The values on g_0, g_1, g_2 are those of Lemma 9.10(i).

(ii) From $[P_g \mid R_g] = \Pi(gz)M_g^{-1}$ and (9.4) one gets $R_{gh}(z) = R_g(hz)R_h(z)$, so r and $\jmath = r^{-1}$ are cocycles. A homomorphism $\Delta \to PSL_2(\mathbb{Z})$ admits several lifts to $SL_2(\mathbb{Z})$ (on the free product $\Delta = \mathbb{Z}/3 * \mathbb{Z}/4$ the signs on the two factors may be chosen independently, subject to the orders); we take the lift determined by $g_1 \mapsto \left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $g_2 \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. the one with $\jmath_{g_2} = +\tau$. This is a homomorphism on $\Delta = \mathbb{Z}/3 * \mathbb{Z}/4$ because the two matrices have orders 3 and 4, and it lifts $\tau \circ g_i = (a_i\tau + b_i)/(c_i\tau + d_i)$. Therefore $(g, z) \mapsto c_g\tau(z) + d_g$ is also a cocycle (by the Δ-equivariance of τ), and it agrees with $\jmath_g$ on the generators g_1, g_2; hence everywhere. The last formula is the usual derivative rule for Möbius transformations: $g^*(d\tau) = d(\tau \circ g) = (c_g\tau + d_g)^{-2}d\tau$.

(iii) Let $z \in \mathfrak{h}_z^\circ$. Then π is a local biholomorphism at z and $\pi(z) \in B^\circ$, so $\frac{d}{dz}(t \circ \pi)(z) \neq 0$ and $1728\, t(\pi(z)) \notin \{0, 1728\}$. By $j \circ \tau = 1728\, t \circ \pi$ (j the modular invariant) the value $\tau(z)$ is not $PSL_2(\mathbb{Z})$-equivalent to ρ or i, so the standard fact $j'(w) \neq 0$ off the $PSL_2(\mathbb{Z})$-orbits of ρ, i gives $j'(\tau(z)) \neq 0$; differentiating $j \circ \tau = 1728\, t \circ \pi$ yields $\dot\tau(z) \neq 0$. $\square$

Step 1: normal form of invariant forms on the covers

Lemma 9.15. Let $U \subseteq \mathfrak{h}_z$ be open and connected and let θ be a holomorphic p-form on $\mathcal{T}|_U$, $p \in \{1, 2, 3\}$; denote again by θ its pullback to $U \times \mathbb{C}^2$. We use the following two facts established earlier: $L(z) = \Pi(z)\Lambda$ is a lattice of full rank in $\mathbb{C}^2$ for *every* $z \in \mathfrak{h}_z$ (Theorem 3.4), and $\dot\tau \neq 0$ on $\mathfrak{h}_z^\circ$ (Lemma 9.14(iii)). Then there are holomorphic functions on U such that

$$\text{(a) } \theta = a(z)\, dz + c(z) \cdot d\zeta, \qquad \text{(b) } \theta = dz \wedge \big(b(z) \cdot d\zeta\big), \qquad \text{(c) } \theta = c(z)\, dz \wedge d\zeta_1 \wedge d\zeta_2$$

for $p = 1, 2, 3$ respectively; moreover in case (a) one has $c(z)\dot Z(z) = 0$ for all $z \in U$.

If in addition U is invariant under a subgroup $\Delta' \subseteq \Delta$ and θ is invariant under $\tilde g$ for all $g \in \Delta'$, then for $g \in \Delta'$:

$$\text{(a)} \quad c(gz)R_g(z) = c(z), \qquad a(gz)\dot g(z) = a(z), \qquad c(gz)\dot R_g(z) = 0; \tag{9.8}$$

$$\text{(b)} \quad b(gz)R_g(z)\dot g(z) = b(z); \tag{9.9}$$

$$\text{(c)} \quad c(gz)\, \dot g(z) \det R_g(z) = c(z). \tag{9.10}$$

Proof. We have $T_\lambda^* dz = dz$ and $T_\lambda^* d\zeta = d\zeta + \dot\Pi(z)\lambda\, dz$.

(a) Write $\theta = a(z, \zeta)dz + c(z, \zeta) \cdot d\zeta$. Then $T_\lambda^*\theta = \big(a \circ T_\lambda + (c \circ T_\lambda)\dot\Pi\lambda\big)dz + (c \circ T_\lambda) \cdot d\zeta$, so T_λ-invariance for all $\lambda \in \Lambda$ gives first $c(z, \zeta + \Pi(z)\lambda) = c(z, \zeta)$; for fixed z the lattice $L(z) = \Pi(z)\Lambda \subset \mathbb{C}^2$ is of full rank (Theorem 3.4), so the entire functions $c_i(z, \cdot)$ are constant: $c = c(z)$. Secondly

$$a(z, \zeta + \Pi(z)\lambda) - a(z, \zeta) = -c(z)\dot\Pi(z)\lambda \qquad (\lambda \in \Lambda), \tag{9.11}$$

whose right-hand side does not depend on ζ; hence each $\partial a / \partial \zeta_i$ is $L(z)$-periodic, so constant in ζ, and $a(z, \zeta) = a(z) + K(z) \cdot \zeta$ with a, K holomorphic on U. Then (9.11) reads

$$K(z)\, \Pi(z)\lambda = -c(z)\dot\Pi(z)\lambda \qquad (\lambda \in \Lambda),$$

an identity between $\mathbb{R}$-linear functions of λ, hence valid for $\lambda \in \Lambda \otimes \mathbb{R}$. Restrict to $\Lambda_{\text{tor}} \otimes \mathbb{R}$: there $\Pi(z)\lambda$ runs through $\mathbb{R}^2 \subset \mathbb{C}^2$ and $\dot\Pi(z)\lambda = 0$, because $\Pi = [Z \mid I]$ and $\Lambda_{\text{tor}} = \langle \hat w, \hat\delta \rangle$. Thus the $\mathbb{C}$-linear form $K(z)$ vanishes on $\mathbb{R}^2$, so $K \equiv 0$; feeding this back, $c(z)\dot\Pi(z)\lambda = 0$ for all $\lambda \in \Lambda$, i.e. $c(z)\dot Z(z) = 0$.

(b) Write $\theta = a(z, \zeta)d\zeta_1 \wedge d\zeta_2 + dz \wedge (b(z, \zeta) \cdot d\zeta)$. With $p := \dot\Pi(z)\lambda$ one has $T_\lambda^*(d\zeta_1 \wedge d\zeta_2) = d\zeta_1 \wedge d\zeta_2 + dz \wedge (-p_2 d\zeta_1 + p_1 d\zeta_2)$, so invariance gives $a \circ T_\lambda = a$, hence $a = a(z)$ as in (a), and

$$b(z, \zeta + \Pi(z)\lambda) - b(z, \zeta) = -a(z)\, (-p_2, p_1).$$

Exactly as above b is affine in ζ with vanishing linear part (restrict to $\Lambda_{\text{tor}} \otimes \mathbb{R}$, where $p = 0$), and then $a(z)\dot Z(z) = 0$. By Lemma 9.14(iii), $\dot\tau \neq 0$ off a discrete set, so $a \equiv 0$ and $\theta = dz \wedge (b(z) \cdot d\zeta)$.

(c) $T_\lambda^*(dz \wedge d\zeta_1 \wedge d\zeta_2) = dz \wedge d\zeta_1 \wedge d\zeta_2$, so the coefficient is $L(z)$-periodic, hence a function of z.

Finally $\tilde g^* dz = \dot g(z)dz$ and $\tilde g^* d\zeta = R_g(z)d\zeta + \dot R_g(z)\zeta\, dz$. In case (a),

$$\tilde g^*\theta = \big(a(gz)\dot g(z) + c(gz)\dot R_g(z)\zeta\big)dz + c(gz)R_g(z) \cdot d\zeta,$$

and comparison of the $d\zeta$-part, and of the $d\zeta$-free part for all ζ, gives (9.8). In case (b), $\tilde g^*\theta = \dot g(z)\, dz \wedge \big(b(gz)R_g(z) \cdot d\zeta\big)$, giving (9.9); in case (c), $\tilde g^*\theta = c(gz)\dot g(z) \det R_g(z)\, dz \wedge d\zeta_1 \wedge d\zeta_2$, giving (9.10). $\square$

By Lemma 9.14(i), writing $c = (c_1, c_2)$ and $b = (b_1, b_2)$, the relations (9.8)–(9.10) say in coordinates:

$$\begin{aligned} &c_2 \circ g = c_2, \qquad (b_2\, dz) \text{ is } \Delta'\text{-invariant,} \\ &\text{if } c_2 \equiv 0: \ c_1(gz) r_g(z) = c_1(z), \\ &\text{if } b_2 \equiv 0: \ g^*(b_1 dz) = \jmath_g\, b_1 dz, \\ &g^*(c\, dz) = \jmath_g\, c\, dz \ \text{ in case (c).} \end{aligned} \tag{9.12}$$

Step 2: matching the charts, and regularity at Σ

Lemma 9.16. Let $\theta \in H^0(X, \Omega_X^p)$, $p \in \{1, 2, 3\}$, and let a, c $(p = 1)$, b $(p = 2)$, c $(p = 3)$ be the coefficients on $\mathfrak{h}_z^\circ$ of the pullback of θ to $\mathfrak{h}_z^\circ \times \mathbb{C}^2$, as in Lemma 9.15 applied to $U = \mathfrak{h}_z^\circ$. Then:

(i) c $(p = 1)$, b $(p = 2)$ and c $(p = 3)$ extend holomorphically to all of $\mathfrak{h}_z$, and the relations

$$c(gz)R_g(z) = c(z), \qquad c(gz)\dot{R}_g(z) = 0 \quad \text{(from (9.8))}, \qquad (9.9), \qquad (9.10)$$

hold on $\mathfrak{h}_z$ for all $g \in \Delta$. If moreover $p = 1$ and $c \equiv 0$, then a extends holomorphically to $\mathfrak{h}_z$ as well and the remaining relation $a(gz)\dot{g}(z) = a(z)$ of (9.8) holds on all of $\mathfrak{h}_z$; in general that relation is asserted only on $\mathfrak{h}_z^\circ$, where a is defined.

(ii) On the cusp neighbourhood U_r with coordinate s ($t_c \circ \pi = e^{2\pi i s}$, $s \circ g_0 = s - 1$) put

$$\tilde{a} := a\frac{dz}{ds}, \qquad \tilde{b} := b\,\frac{dz}{ds}, \qquad \tilde{c} := c\,\frac{dz}{ds} \quad (p = 3),$$

so that $\theta = \tilde{a}\, ds + c \cdot d\zeta$, $\theta = ds \wedge (\tilde{b} \cdot d\zeta)$, $\theta = \tilde{c}\, ds \wedge d\zeta_1 \wedge d\zeta_2$ respectively. Then $c, \tilde{a}, \tilde{b}, \tilde{c}$ are g_0-invariant, hence holomorphic functions of t_c on $0 < |t_c| < r$, and

$$\begin{aligned} &c(t_c) = O(t_c), \quad \tilde{a}(t_c) = O(t_c) \quad (p = 1); \\ &\tilde{b}(t_c) = O(t_c) \quad (p = 2); \\ &\tilde{c}(t_c) = O(t_c) \quad (p = 3), \end{aligned}$$

where $O(t_c)$ means: extends holomorphically across $t_c = 0$ and vanishes there.

Proof. (i) Regularity at p_1, p_2. By §5 the quotient map $J_j' = \mathcal{T}|_{\Delta_j} \to N_j$ by $\langle \tilde{g}_j^{\log} \rangle$ is étale, so $\theta|_{N_j}$ pulls back to a holomorphic p-form on J_j' and hence to a Λ-invariant p-form on $\Delta_j \times \mathbb{C}^2$. By Lemma 9.15 (with $U = \Delta_j$) its coefficients $a^{(j)}, c^{(j)}, b^{(j)}$ are holomorphic on all of Δ_j. Over D_j^* the two descriptions of θ are compared by the regluing map h_j of §5, translation by the section $\sigma_j(z) = \frac{\log s_j}{2\pi i}\Pi(z)v_j$, for which

$$h_j^* dz = dz, \qquad h_j^* d\zeta = d\zeta + \dot{\sigma}_j(z)\, dz.$$

Hence $h_j^*\big(a\, dz + c \cdot d\zeta\big) = (a + c \cdot \dot{\sigma}_j)dz + c \cdot d\zeta$, $h_j^*\big(dz \wedge (b \cdot d\zeta)\big) = dz \wedge (b \cdot d\zeta)$ and $h_j^*\big(c\, dz \wedge d\zeta_1 \wedge d\zeta_2\big) = c\, dz \wedge d\zeta_1 \wedge d\zeta_2$: in all three cases the coefficients c, b, c of Lemma 9.15 are unchanged; in particular $h_j^* e = e$ for the relative 2-form $e = d\zeta_1 \wedge d\zeta_2$. Therefore $c = c^{(j)}$, $b = b^{(j)}$ on Δ_j^*, and these functions extend holomorphically across z_j; the same holds at every point of the Δ-orbit of z_j by the relations for c, b. If $p = 1$ and $c \equiv 0$, then h_j^* is the identity on such forms and $a = a^{(j)}$ extends likewise. The stated relations persist on $\mathfrak{h}_z$ (resp., for a when $c \not\equiv 0$, on $\mathfrak{h}_z^\circ$) by continuity.

(ii) The cusp. Since $R_{g_0} = I$, $\jmath_{g_0} = 1$ and $s \circ g_0 = s - 1$, the relations (9.8)–(9.10) for $g = g_0$ say precisely that $c, \tilde{a}, \tilde{b}, \tilde{c}$ are invariant under $s \mapsto s - 1$; so they are holomorphic functions of $t_c = e^{2\pi i s}$ on the punctured disc. The gluing of J and N_0 over D_0^* is tautological via

$$\begin{aligned} &E_0(z, \zeta) = \big(e^{2\pi i \zeta_1}, e^{2\pi i \zeta_2}, e^{2\pi i s(z)}\big) = (x_1, x_2, t), \\ &d\zeta_k = \frac{1}{2\pi i}\frac{dx_k}{x_k}, \qquad ds = \frac{1}{2\pi i}\frac{dt}{t}, \end{aligned}$$

and $Y_\varepsilon \to N_0 = Y_\varepsilon / \{\Psi_{\bar{\lambda}}\}$ is étale; so θ pulls back to a holomorphic p-form Θ on Y_ε whose restriction to $Y_\varepsilon \cap T_{N'}$ is given by the above substitution.

Fix a maximal cone $\sigma \in \mathfrak{F}$, the cone over a triangle of $\mathcal{T}_{A_2}$ with vertices $v_0, v_1, v_2 \in \mathbb{Z}^2$. The generators $u_k := (v_k, 1)$ form a $\mathbb{Z}$-basis of N' (the triangles of $\mathcal{T}_{A_2}$ are unimodular), so $U_\sigma \cong \mathbb{C}^3$ with coordinates

$w_k := \chi^{m^k}$, (m^k) the dual basis, and $\chi^m = \prod_k w_k^{\langle m, u_k \rangle}$ for all $m \in M'$. In particular

$$t = w_0 w_1 w_2, \qquad x_i = \prod_k w_k^{(v_k)_i}, \qquad \frac{dt}{t} = \sum_k \frac{dw_k}{w_k}, \qquad \frac{dx_i}{x_i} = \sum_k (v_k)_i \frac{dw_k}{w_k}.$$

Substituting:

$$\begin{aligned} p = 1: \quad \Theta &= \frac{1}{2\pi i} \sum_{k=0}^{2} \left(\tilde{a}(t) + c(t) \cdot v_k \right) \frac{dw_k}{w_k}, \\ p = 2: \quad \Theta &= \frac{1}{(2\pi i)^2} \sum_{k<l} \left(\tilde{b}(t) \cdot (v_l - v_k) \right) \frac{dw_k \wedge dw_l}{w_k w_l}, \\ p = 3: \quad \Theta &= \pm \frac{1}{(2\pi i)^3} \frac{\tilde{c}(t)}{t} \, dw_0 \wedge dw_1 \wedge dw_2, \end{aligned}$$

where $c \cdot v := c_1 v^{(1)} + c_2 v^{(2)}$ and the sign in the last line comes from $\det(u_0, u_1, u_2) = \pm 1$.

Now use the coordinate lines and planes of U_σ, which lie in Y_ε where $|t| < \varepsilon$. For $p = 1$ restrict Θ to $\{w_l = 1,\ l \neq k\}$, a disc with coordinate $w_k = t$: the restriction is $\frac{1}{2\pi i}(\tilde{a}(t) + c(t) \cdot v_k)\frac{dt}{t}$, holomorphic at $t = 0$, so $\tilde{a} + c \cdot v_k = O(t)$ for $k = 0, 1, 2$. Subtracting, $c(t) \cdot (v_k - v_0) = O(t)$ for $k = 1, 2$; as $v_1 - v_0, v_2 - v_0$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, $c = O(t)$ and then $\tilde{a} = O(t)$. For $p = 2$ restrict to $\{w_n = 1\}$ for the third index $n \notin \{k, l\}$, a bidisc with $t = w_k w_l$: holomorphy at $w_k = 0$ (with $w_l = 1$, so $t = w_k$) gives $\tilde{b}(t) \cdot (v_l - v_k) = O(t)$ for all pairs, whence $\tilde{b} = O(t)$ as before. For $p = 3$ no restriction of the *form* is taken: since $dw_0 \wedge dw_1 \wedge dw_2$ is a frame of Ω^3 on the chart $U_\sigma \cong \mathbb{C}^3$ and Θ is holomorphic on $U_\sigma \cap Y_\varepsilon$, the coefficient of Θ relative to this frame, namely $\pm(2\pi i)^{-3}\, \tilde{c}(t)/t$ with $t = w_0 w_1 w_2$, is a holomorphic *function* on $U_\sigma \cap Y_\varepsilon$; evaluating this function along $\{w_1 = w_2 = 1\}$, where $t = w_0$, shows that $\tilde{c}(t)/t$ is holomorphic at $t = 0$, i.e. $\tilde{c} = O(t)$. ☐

Step 3: descent through the elliptic points, and two vanishing lemmas

Lemma 9.17. Let $j \in \{1, 2\}$, $m = m_j$, $\zeta = \zeta_j$, and let $s = s_j$ be the linearising coordinate on Δ_j ($s \circ g_j = \zeta s$, $t_j \circ \pi = s^m$).

(D0) If C is holomorphic on Δ_j with $C \circ g_j = C$, then C is a holomorphic function of t_j.

(D1) If $\alpha = A(s)\, ds$ is a holomorphic 1-form on Δ_j with $g_j^* \alpha = \alpha$, then $A = \sum_{n \equiv -1\,(m)} A_n s^n$ and $\alpha = \sum_{l \geq 1} \frac{A_{lm-1}}{m}\, t_j^{l-1} dt_j$; in particular α is the pullback of a holomorphic 1-form on D_j.

(D2) If $\kappa = C(s)\, ds^{\otimes 3}$ is a holomorphic cubic differential on Δ_j with $g_j^* \kappa = \kappa$, then $C = \sum_{n \equiv -3\,(m)} C_n s^n$ and κ is the pullback of a meromorphic cubic differential on D_j with a pole of order at most 2 at p_j.

Proof. $g_j^* ds = \zeta\, ds$, so the three invariance conditions read $C(\zeta s) = C(s)$, $A(\zeta s)\zeta = A(s)$, $C(\zeta s)\zeta^3 = C(s)$; comparing Taylor coefficients gives $n \equiv 0$, $n \equiv -1$, $n \equiv -3 \pmod{m}$ respectively. With $t = s^m$, $dt = m s^{m-1} ds$: for (D1), $s^{lm-1} ds = t^{l-1} dt/m$; for (D2) with $m = 3$ and $n = 3l$, $s^{3l} ds^{\otimes 3} = t^l\, dt^{\otimes 3}/(27 s^6) = t^{l-2} dt^{\otimes 3}/27$; with $m = 4$ and $n = 4l + 1$, $s^{4l+1} ds^{\otimes 3} = s^{4l+1} dt^{\otimes 3}/(64 s^9) = t^{l-2} dt^{\otimes 3}/64$. In both cases $l \geq 0$, so the order at p_j is ≥ -2. ☐

Lemma 9.18 (weight 0)**.** Let α be a Δ-invariant holomorphic 1-form on $\mathfrak{h}_z$ such that, on U_r, $\alpha = \tilde{a}(s)\, ds$ with $\tilde{a} = O(t_c)$ in the sense of Lemma 9.16(ii). Then $\alpha = 0$.

Proof. Over B° the map π is an unramified Δ-covering, so α descends to a holomorphic 1-form α_B on B°; by (D1) applied at z_1, z_2 (the stabiliser of z_j in Δ being $\langle g_j \rangle$), α_B extends holomorphically over p_1 and p_2. At p_0, $\alpha = \tilde{a}(t_c) \frac{dt_c}{2\pi i\, t_c}$ and $\tilde{a} = O(t_c)$, so α_B extends holomorphically over p_0 as well. Hence $\alpha_B \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$ and $\alpha = \pi^* \alpha_B = 0$. ☐

Lemma 9.19 (weight $+1$)**.** Let ψ be a holomorphic 1-form on $\mathfrak{h}_z$ with

$$g^* \psi = \jmath_g\, \psi \qquad (g \in \Delta)$$

and such that, on U_r, $\psi = \tilde{b}(s)\, ds$ with $\tilde{b} = O(t_c)$. Then $\psi = 0$.

Proof. Consider the holomorphic cubic differential on $\mathfrak{h}_z$

$$\kappa := \psi \otimes \psi \otimes d\tau.$$

By Lemma 9.14(ii), $g^*\kappa = \jmath_g^2 \cdot \jmath_g^{-2}\,\kappa = \kappa$ for all $g \in \Delta$, i.e. κ is Δ-invariant. As in Lemma 9.18 it descends over B° to a holomorphic cubic differential κ_B on B°, and by (D2) at z_1, z_2 it extends meromorphically over p_1, p_2 with poles of order at most 2.

At p_0: write $\psi = \tilde{b}\, ds = t_c\, r(t_c) \frac{dt_c}{2\pi i\, t_c} = \frac{r(t_c)}{2\pi i}\, dt_c$ with r holomorphic at 0, and (§3) $\tau = s + h(t_c)$, so

$$d\tau = ds + h'(t_c)\, dt_c = \Big(\frac{1}{2\pi i\, t_c} + h'(t_c)\Big) dt_c.$$

Hence

$$\kappa_B = \frac{r(t_c)^2}{(2\pi i)^2}\Big(\frac{1}{2\pi i\, t_c} + h'(t_c)\Big)\, dt_c^{\otimes 3}$$

has a pole of order at most 1 at p_0 (recall t_c is a local coordinate there). Therefore

$$\begin{aligned} \kappa_B &\in H^0\big(\mathbb{P}^1, (\Omega^1_{\mathbb{P}^1})^{\otimes 3}(2p_1 + 2p_2 + p_0)\big) \\ &= H^0\big(\mathbb{P}^1, \mathcal{O}(-6+5)\big) = H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0. \end{aligned}$$

So $\kappa = 0$. Writing $\psi = b_1(z)\, dz$ this means $b_1^2\, \dot\tau \equiv 0$; since $\dot\tau \neq 0$ on $\mathfrak{h}_z^\circ$ (Lemma 9.14(iii)) we get $b_1 \equiv 0$ on $\mathfrak{h}_z^\circ$, hence $\psi = 0$. $\square$

Step 4: the vanishing theorem

Proposition 9.20. $H^0(X, \Omega^p_X) = 0$ for $p = 1, 2, 3$; that is, $h^{1,0} = h^{2,0} = h^{3,0} = 0$.

Proof. In each case we show that the pullback of the form to $\mathfrak{h}_z^\circ \times \mathbb{C}^2$ vanishes; since $J|_{B^\circ}$ is dense in the connected manifold X, the form itself is then 0.

$p = 1$. Let $\omega \in H^0(X, \Omega^1_X)$, with coefficients a, c as in Lemmas 9.15 and 9.16; thus c is holomorphic on $\mathfrak{h}_z$, $c(gz)R_g(z) = c(z)$, $c(z)\dot{Z}(z) = 0$, and $c = O(t_c)$ at the cusp.

By Lemma 9.14(i) the second component of $c(gz)R_g(z) = c(z)$ reads $c_2 \circ g = c_2$: the function c_2 is Δ-invariant and holomorphic on $\mathfrak{h}_z$. By (D0) it is a holomorphic function of t_j near p_j ($j = 1, 2$), and it is a holomorphic function of t on B°; hence it descends to a holomorphic function on $B \smallsetminus \{p_0\} \cong \mathbb{C}$. By Lemma 9.16(ii) it extends holomorphically across p_0 with value 0 there. A global holomorphic function on $\mathbb{P}^1$ is constant, so $c_2 \equiv 0$.

The second column of $c\,\dot{Z} = 0$ now gives $c_1\dot\tau + c_2\dot\mu = c_1\dot\tau = 0$, and $\dot\tau \neq 0$ on $\mathfrak{h}_z^\circ$, so $c_1 \equiv 0$ on $\mathfrak{h}_z^\circ$ and hence on $\mathfrak{h}_z$. Thus $c \equiv 0$ and $\omega = a(z)\, dz$ upstairs, with a holomorphic on all of $\mathfrak{h}_z$ by Lemma 9.16(i) and $a(gz)\dot{g}(z) = a(z)$ on $\mathfrak{h}_z$ by the same lemma; i.e. $\alpha := a\, dz$ is a Δ-invariant holomorphic 1-form on $\mathfrak{h}_z$, and $\tilde{a} = O(t_c)$ at the cusp by Lemma 9.16(ii). Lemma 9.18 gives $\alpha = 0$, i.e. $\omega = 0$.

$p = 2$. Let $\eta \in H^0(X, \Omega^2_X)$; upstairs $\eta = dz \wedge (b(z) \cdot d\zeta)$ with b holomorphic on $\mathfrak{h}_z$ (Lemmas 9.15, 9.16) and $b(gz)R_g(z)\dot{g}(z) = b(z)$.

The second component of this relation is $b_2(gz)\dot{g}(z) = b_2(z)$ (Lemma 9.14(i)), i.e. $\alpha := b_2\, dz$ is a Δ-invariant holomorphic 1-form on $\mathfrak{h}_z$; by Lemma 9.16(ii) its cusp coefficient $\tilde{b}_2$ is $O(t_c)$. Lemma 9.18 gives $b_2 \equiv 0$.

The first component then reads $b_1(gz)r_g(z)\dot{g}(z) = b_1(z)$, i.e. $\psi := b_1\, dz$ satisfies $g^*\psi = \jmath_g\psi$ for all $g \in \Delta$; and $\tilde{b}_1 = O(t_c)$ at the cusp. Lemma 9.19 gives $\psi = 0$, so $b \equiv 0$ and $\eta = 0$.

$p = 3$. Let $\Omega \in H^0(X, \Omega^3_X)$; upstairs $\Omega = c(z)\, dz \wedge d\zeta_1 \wedge d\zeta_2$ with c holomorphic on $\mathfrak{h}_z$ and, by (9.10) and $\det R_g = r_g$,

$$c(gz)\dot{g}(z)\, r_g(z) = c(z), \qquad \text{i.e.} \qquad g^*\psi = \jmath_g\, \psi \ \text{ for } \ \psi := c\, dz.$$

By Lemma 9.16(ii), $\tilde{c} = O(t_c)$ at the cusp, so Lemma 9.19 gives $\psi = 0$ and $\Omega = 0$. $\square$

This proves Theorem 9.1(vi). For $p = 3$ the vanishing agrees with $h^{3,0} = h^{0,3} = 0$ from Proposition 9.13 and Serre duality (equivalently, with the non-effectivity of $K_X = f^*\mathcal{O}(-1) \otimes \mathcal{O}(2S_2)$, Proposition 9.11); the cases $p = 1, 2$ are new. The same arguments apply verbatim to the variant X' of §6.

9.6 The remaining Hodge numbers

Remark 9.21 (What is proved, and the expected values). Combining Proposition 9.13, Proposition 9.20, $\chi(\Omega^1_X) = -1$, $\chi(\Omega^2_X) = 1$ (§9.4) and Serre duality $h^{p,q} = h^{3-p,3-q}$, the Hodge numbers of X are

$$h^{0,0} = h^{3,3} = h^{0,1} = h^{3,2} = 1, \qquad h^{p,0} = 0 \ (p \geq 1), \quad h^{0,q} = 0 \ (q \geq 2),$$

hence also $h^{3,q} = 0$ for $q \leq 1$ and $h^{p,3} = 0$ for $p \leq 2$; and

$$\chi(\Omega^1_X) = h^{1,0} - h^{1,1} + h^{1,2} - h^{1,3} = -h^{1,1} + h^{1,2} = -1,$$
$$\chi(\Omega^2_X) = h^{2,0} - h^{2,1} + h^{2,2} - h^{2,3} = -h^{2,1} + h^{2,2} = 1,$$

the two relations being equivalent under Serre duality. Thus *every* Hodge number of X is determined by the single integer

$$h^{1,2} = h^{2,1} \ \geq 0, \qquad \text{via} \qquad h^{1,1} = h^{2,2} = h^{1,2} + 1.$$

The remaining Hodge numbers are determined by $h^{1,2}$, which we do not compute; the expected value is

$$h^{1,2} = h^{2,1} = 1, \qquad \text{whence} \qquad h^{1,1} = h^{2,2} = 2.$$

(Equivalently: $\dim_{\mathbb{C}} E_1$ of the Frölicher spectral sequence equals $6 + 4h^{1,2}$, expected 10.) A proof of $h^{1,2} = 1$ would require the direct images $R^q f_* \Omega^1_X(\log D)$, $D := W + S_1 + S_2$, equivalently those of $E := \Omega^1_X(\log D)/f^*\Omega^1_B(\log \Sigma)$, at the three singular fibres, where $\Omega^1_{X/B}$ has torsion along $W \cup S_1 \cup S_2$ and the invariant-theoretic description on the covers J'_j and on $Y_{\mathfrak{F}}$ must be matched with the residue sequences.

Corollary 9.22. The Frölicher spectral sequence of X does not degenerate at E_1. Moreover $\partial\colon H^{0,1} \to H^{1,1}$ is injective, so $h^{1,1} \geq 1$.

Proof. By Proposition 9.13, $h^{0,1} = 1$, and $h^{1,0} = 0$ by Proposition 9.20. If the spectral sequence degenerated at E_1, then $b_1(X) = h^{1,0} + h^{0,1} = 1$. But $b_1(X) = 0$: indeed X is simply connected by the van Kampen computation of §7 (a computation with the fundamental groups of the four pieces J, N_1, N_2, N_0 and of their overlaps). This contradiction proves the first assertion.

For the second, the generator of $H^{0,1}$ must die in the spectral sequence, since $E^{0,1}_\infty$ is a subquotient of $H^1(X;\mathbb{C}) = 0$. No differential enters $E^{0,1}_r$ (the groups $E^{-r,r}_r$ vanish). The differentials leaving it are $d_1 = \partial\colon H^{0,1} \to H^{1,1}$; $d_2\colon E^{0,1}_2 \to E^{2,0}_2$, whose target is a subquotient of $H^{2,0} = 0$ (Proposition 9.20); and d_r for $r \geq 3$, with target $E^{r,2-r}_r = 0$ since $2 - r < 0$. Hence the class must be killed by d_1, i.e. ∂ is injective on $H^{0,1}$, and $h^{1,1} \geq h^{0,1} = 1$. □

9.7 The vertical $\mathbb{C}^*$-action

Proposition 9.23. $h^0(X, T_X) = 1$ and $\mathrm{Aut}^0(X) = \mathbb{C}^*$, generated by the vertical field ξ determined by $\hat{\delta} \in \Lambda^{\langle A_1, A_2\rangle} = \mathbb{Z}\hat{\delta}$; every automorphism in $\mathrm{Aut}^0(X)$ preserves each fibre of f.

Proof. Construction. In the flat coordinates of $\mathcal{T}$ let ξ be the constant vertical field $e_2 = \Pi(z)\hat{\delta}$; it is Λ-invariant and, by (9.4) and $A_g\hat{\delta} = \hat{\delta}$, invariant under all $\tilde{g}$, so it descends to J. Its flow is translation by ςe_2 and factors through $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$ since $e_2 \in L(z)$. These translations commute with $\tilde{g}^{\log}_j$ (whose linear part fixes e_2), so ξ extends to N_1, N_2; under E_0 it is the one-parameter subgroup of $T_{N'}$ with cocharacter $\hat{\delta} \leftrightarrow e_2 \in \Lambda_{\mathrm{tor}}$, acting algebraically on $Y_{\mathfrak{F}}$ and commuting with all $\Psi_{\bar{\lambda}}$. So ξ generates an effective fibrewise $\mathbb{C}^*$-action on X.

Verticality. For $v \in H^0(T_X)$ write $df(v) = \varphi\,\partial_t$ with $\varphi \in H^0(\mathcal{O}(2))$. Over D_j pull v back along the étale map $q_j\colon J'_j \to N_j$ of §5; since $f \circ q_j = \pi_j \circ \tilde{f}$ with $\pi_j(s) = s^{m_j} = t$, the resulting field on Δ_j has horizontal part

$$\varphi(s^{m_j})\, m_j^{-1} s^{1-m_j} \partial_s,$$

which is holomorphic only if $\varphi(p_j) = 0$; at a triple point of W, $t = z_1 z_2 z_3$ and $df(v) = \sum a_i z_j z_k$ vanishes, so $\varphi(p_0) = 0$. Hence $\varphi \in H^0(\mathcal{O}(2)(-p_0 - p_1 - p_2)) = 0$.

Uniqueness. Vertical fields are the sections of $\mathcal{V} := f_* T_{X/B}$, which we show to be locally free of rank 2 with $\mathcal{O}\xi \subset \mathcal{V}$ a subbundle.

Over B°, $\mathcal{V}$ consists of the holomorphic $v\colon \mathfrak{h}_z^\circ \to \mathbb{C}^2$ with $v \circ g = R_g v$ (the fibrewise-constant fields on a torus), and near p_j the same description holds on Δ_j, the translation part of $\tilde{g}_j^{\log}$ not acting on constant fibre fields; so there $\mathcal{V} = \big(\mathcal{O}(\Delta_j)^2\big)^{\mathbb{Z}/m_j}$ for the action $v \mapsto R_{g_j}^{-1}(v \circ g_j)$, whose invariants are exactly the v with $v \circ g_j = R_{g_j} v$.

Near p_0 we compute in the toric charts of §4. In $U_\sigma \cong \mathbb{C}^3$ with coordinates w_0, w_1, w_2 as in the proof of Lemma 9.16(ii), the field attached to $n \in \mathbb{Z}^2 \subset N'$ is $\partial_n = \sum_k \langle m^k, (n,0)\rangle\, w_k \partial_{w_k}$, holomorphic on U_σ and annihilating $t = w_0 w_1 w_2$; for $n = e_1, e_2$ these are the fields $x_1\partial_{x_1}, x_2\partial_{x_2}$ of the open orbit, and they are $\Psi_{\bar\lambda}$-invariant (the shears act by elements of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ fixing Λ_{tor} up to the translation part, which does not act on constant torus fields) — hence they are sections of $\mathcal{V}$ over D_0. Conversely, a section of $\mathcal{V}$ over D_0^* is fibrewise constant, so of the form $a_1(t)x_1\partial_{x_1} + a_2(t)x_2\partial_{x_2}$ with a_i holomorphic on D_0^*; in the chart U_σ it reads $\sum_k c_k(t)\, w_k \partial_{w_k}$ with $c_k := a_1(m^k)_1 + a_2(m^k)_2$. Restrict to the line $\{w_1 = w_2 = 1\}$, on which $t = w_0$: the coefficient of ∂_{w_0} is $w_0 c_0(t) = t\, c_0(t)$, while the coefficients of $\partial_{w_1}, \partial_{w_2}$ are $c_1(t)$ and $c_2(t)$. Holomorphy of the field therefore forces c_1 and c_2 to extend holomorphically across $t = 0$ (for $k = 0$ only $t\, c_0$ is controlled, and we do not use it). Now the restrictions of m^0, m^1, m^2 to $\mathbb{Z}^2 \times \{0\}$ satisfy the single relation $m^0 + m^1 + m^2|_{\mathbb{Z}^2\times\{0\}} = 0$ (a vector $(n,0)$ has height-0 coordinates with respect to u_0, u_1, u_2) and span the dual of $\mathbb{Z}^2$; hence any two of them form a basis, and in particular the two vectors $\big((m^k)_1, (m^k)_2\big)$, $k = 1,2$, are linearly independent. Therefore a_1, a_2 themselves extend holomorphically across p_0, and $\mathcal{V}$ is free near p_0 with frame $x_1\partial_{x_1}, x_2\partial_{x_2}$, i.e. (e_1, e_2).

The line $\mathbb{C}e_2$ is Δ-invariant, so $\mathcal{O}\xi \subset \mathcal{V}$ with quotient $\mathcal{Q}$ carrying the multivalued frame $\bar{e}_1$. By Lemma 9.10 the e_1-component satisfies $v_1 \circ g = (\det R_g)\, v_1$, so the frames $\bar{e}_1$ of $\mathcal{Q}$ and e of $\mathfrak{K}$ have reciprocal automorphy factors. At p_0 both $\bar{e}_1$ and e generate, by the previous paragraph and Proposition 9.11. At p_j, lifting to Δ_j and using $s \circ g_j = \zeta_j s$ together with $\det R_{g_j}(z_j) = \zeta_j^{a_j}$, a holomorphic invariant v_1 is s^{c_j} times a unit with $c_j \in \{0, \dots, m_j - 1\}$, $c_j \equiv a_j \pmod{m_j}$, i.e. $(c_1, c_2) = (2,1)$; thus $s^{c_j}\bar{e}_1$ is a frame of $\mathcal{Q}$ at p_j — in particular $\mathcal{Q}$ is locally free of rank 1 there and $\mathcal{O}\xi \subset \mathcal{V}$ is a subbundle — and $\bar{e}_1$ has exponent $-a_j/m_j$ where e has $+a_j/m_j$. Hence $\mathcal{Q} \cong \mathfrak{K}^{-1} \cong \mathcal{O}(-1)$, $h^0(\mathcal{Q}) = 0$, so $H^0(\mathcal{V}) = \mathbb{C}\xi$ and $h^0(T_X) = 1$.

The group $\mathrm{Aut}^0(X)$. Since $h^0(T_X) = 1$, the group $\mathrm{Aut}^0(X)$ is a connected complex Lie group of dimension 1 with Lie algebra $\mathbb{C}\xi$; hence the flow of ξ gives a surjective homomorphism $\mathbb{C} \to \mathrm{Aut}^0(X)$, $\varsigma \mapsto \varphi_\varsigma$, and $\mathrm{Aut}^0(X) = \mathbb{C}/P$ with $P := \{\varsigma \in \mathbb{C} : \varphi_\varsigma = \mathrm{id}_X\}$ discrete. A priori P has rank 0, 1 or 2, i.e. $\mathrm{Aut}^0(X) \cong \mathbb{C}$, $\mathbb{C}^*$ or an elliptic curve; we show $P = \mathbb{Z}$, which excludes both the first case ($P = 0$) and the third case (P of rank 2, $\mathrm{Aut}^0(X)$ compact). Let $\varsigma \in P$. On the smooth fibre $F_{\pi(z)} = \mathbb{C}^2/L(z)$, $z \in \mathfrak{h}_z^\circ$, the map φ_ς is the translation by ςe_2, so $\varsigma e_2 \in L(z) = \Pi(z)\Lambda$ for every $z \in \mathfrak{h}_z^\circ$; as Λ is discrete, Π depends continuously on z and $\mathfrak{h}_z^\circ$ is connected, there is a *single* $\lambda = a\hat\gamma + b\hat u + c\hat w + d\hat\delta \in \Lambda$ with $\Pi(z)\lambda = \varsigma e_2$ for all $z \in \mathfrak{h}_z^\circ$. Its first component gives $6a\mu + b\tau + c \equiv 0$ on $\mathfrak{h}_z$, whence $a = b = c = 0$ by Lemma 9.3, and the second gives $\varsigma = d \in \mathbb{Z}$. Conversely φ_1 is the translation by $e_2 = \Pi(z)\hat\delta \in L(z)$ on each fibre over B°, i.e. the identity on the dense open set $f^{-1}(B^\circ)$ of the connected manifold X, hence $\varphi_1 = \mathrm{id}_X$ and $\mathbb{Z} \subseteq P$. Therefore $P = \mathbb{Z}$, a subgroup of rank 1 contained in a real line — so $\mathrm{Aut}^0(X)$ is neither $\mathbb{C}$ nor an elliptic curve — and

$$\mathrm{Aut}^0(X) = \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*;$$

in particular $\mathrm{Aut}^0(X)$ is not compact, and the $\mathbb{C}^*$-action constructed above is effective. $\square$

Proposition 9.24. The fixed locus of $\mathrm{Aut}^0(X) = \mathbb{C}^*$ is a single double curve $\mathsf{D}_0 \subset W$, $\mathsf{D}_0 \cong \mathbb{P}^1$, containing both triple points of W; in the notation of §4 one has $\mathsf{D}_0 = \nu(C_{e_2})$, the double curve that is the image of the edges of direction $e_2 \leftrightarrow \hat\delta$ — D_2 in the labelling of Proposition 4.6(iii), D_3 in the labelling $\rho_1, \dots, \rho_6$ of Lemma 9.12. The normal weights are $(+1, -1)$. In particular $e(X^{\mathbb{C}^*}) = 2 = e(X)$.

Proof. Fixed points and étale quotients. Each piece of X is of the form U/Γ with $U \to U/\Gamma$ an étale quotient by a group Γ acting freely and properly discontinuously, and with the $\mathbb{C}$-flow Φ_ς of ξ lifted to U and commuting with Γ; explicitly the pairs (U, Γ) are

$$\big(\mathfrak{h}_z^\circ \times \mathbb{C}^2,\ \Lambda \rtimes \Delta\big), \qquad \big(J_j',\ \langle \tilde{g}_j^{\log}\rangle\big)\ (j = 1,2), \qquad \big(Y_\varepsilon,\ \{\Psi_{\bar\lambda}\}\big),$$

with quotients J, N_j, N_0 respectively (recall $J_j' = \mathcal{T}|_{\Delta_j} = (\Delta_j \times \mathbb{C}^2)/\Lambda$). On $\mathfrak{h}_z^\circ \times \mathbb{C}^2$ only the $\mathbb{C}$-flow lifts, and not a $\mathbb{C}^*$-action: the time-1 map is the translation $T_{\hat{\delta}} \in \Lambda$, not the identity. On J'_j and on Y_ε the lifted flow does factor through $\mathbb{C}^*$. All that is used below is that Φ_ς commutes with Γ. If $[y] \in U/\Gamma$ is fixed by the whole group, then for each $\varsigma \in \mathbb{C}$ there is a unique $\gamma_\varsigma \in \Gamma$ with $\Phi_\varsigma(y) = \gamma_\varsigma(y)$; the map $\varsigma \mapsto \gamma_\varsigma$ is continuous into the discrete group Γ and $\gamma_0 = \mathrm{id}$, so $\gamma_\varsigma = \mathrm{id}$ for all ς. Hence the fixed locus downstairs is exactly the image of the set of points of U fixed by Φ_ς for *all* ς.

On $\mathcal{T}$ and on J'_j the flow Φ_ς is the translation by ςe_2 on each fibre torus. Since $e_2 \neq 0$ and $L(z)$ is a discrete subgroup of $\mathbb{C}^2$, the set $\{\varsigma \in \mathbb{C} : \varsigma e_2 \in L(z)\}$ is a discrete subgroup of $\mathbb{C}$; in particular there are ς with $\varsigma e_2 \notin L(z)$, for which the translation is fixed-point free. So no point of those pieces is fixed by the flow for all ς (for special z one may well have $\varsigma e_2 \in L(z)$ for some $\varsigma \notin \mathbb{Z}$, but never for all ς), whence $X^{\mathbb{C}^*} \subset W$.

The fixed locus in W. On $Y_{\mathfrak{F}}$ the action is the one-parameter subgroup of $T_{N'}$ with cocharacter e_2, and an orbit O_σ, $\sigma \in \mathfrak{F}$, is fixed pointwise iff $e_2 \in \mathrm{span}_{\mathbb{R}}(\sigma)$. The cone over a vertex v has span $\mathbb{R}(v,1) \not\ni (e_2, 0)$; the cone over an edge $[v, v+d]$ has span $\langle (v,1), (d,0) \rangle$, containing $(e_2, 0)$ iff $d = \pm e_2$; and e_2 *is* an edge direction of the A_2-triangulation (directions $\pm e_1, \pm e_2, \pm(e_1 - e_2)$). A three-dimensional cone spans $\mathbb{R}^3$, so its orbit — a point — is fixed; but every triangle of $\mathcal{T}_{A_2}$ has an edge of direction e_2 (for $\{v, v+e_1, v+e_2\}$ the edge $[v, v+e_2]$, for $\{v+e_1, v+e_2, v+e_1+e_2\}$ the edge $[v+e_1, v+e_1+e_2]$), so each such point lies in the closure of the fixed curve of that edge and contributes nothing new. Hence the fixed locus in $t^{-1}(0)$ is the union of the closures of the one-dimensional orbits of the edges $[v, v+e_2]$, $v \in \mathbb{Z}^2$. The shears act on the height-1 lattice by the translations $B_0\bar{\lambda}$, which exhaust $\mathbb{Z}^2$ as B_0 is unimodular; hence these curves form one orbit, with image in W a single double curve $\mathsf{D}_0 \cong \mathbb{P}^1$ through P and Q (§4).

Weights. For $\sigma = \mathrm{cone}\{(0,0,1), (0,1,1)\}$ complete to the basis $u_1 = (0,0,1)$, $u_2 = (0,1,1)$, $u_3 = (1,0,0)$ of N', with dual vectors $m_1 = (0,-1,1)$, $m_2 = (0,1,0)$; the normal coordinates of O_σ in $U_\sigma \cong \mathbb{C}^2 \times \mathbb{C}^*$ are m_1, m_2, on which $e_2 = (0,1,0)$ acts with weights $\langle m_1, e_2 \rangle = -1$, $\langle m_2, e_2 \rangle = +1$. Finally $e(X^{\mathbb{C}^*}) = e(\mathbb{P}^1) = 2 = e(X)$, as must hold for a $\mathbb{C}^*$-action. □

Remark 9.25. For every integer $n \geq 2$ the finite subgroup $\mu_n \subset \mathbb{C}^* = \mathrm{Aut}^0(X)$ has the same fixed-point set as $\mathbb{C}^*$ itself,

$$X^{\mu_n} = X^{\mathbb{C}^*} = \mathsf{D}_0 \cong S^2.$$

Indeed, off W the circle $S^1 \subset \mathbb{C}^*$ acts freely: on X° by Lemma 7.21, and on the smooth fibres of N_0 it acts by the translations ςe_2 ($\varsigma \in \mathbb{R} \setminus \mathbb{Z}$), which have no fixed points since $\hat{\delta}$ is primitive in Λ. On W, an element of order n in the one-parameter subgroup of $T_{N'}$ with cocharacter e_2 fixes the orbit of a cone $\hat{\sigma} \in \mathfrak{F}$ pointwise if and only if $\frac{1}{n}(e_2, 0) \in \mathrm{span}_{\mathbb{R}}(\hat{\sigma}) + N'$. For the cone over a vertex v this would force $\frac{1}{n}e_2 \in \mathbb{Z}v + \mathbb{Z}^2$, which is impossible; for the cone over an edge $[v, v+\rho]$ it holds if and only if $\frac{1}{n}e_2 \in \mathbb{R}\rho + \mathbb{Z}^2$, i.e. if and only if $n \mid \det(\rho, e_2)$; and the point orbits of the triangles lie on the closures of the edge curves. Since the edge directions of $\mathcal{T}_{A_2}$ are $\pm e_1, \pm e_2, \pm(e_1 - e_2)$, with $\det(e_1, e_2) = \det(e_1 - e_2, e_2) = 1$ and $\det(e_2, e_2) = 0$, for every n exactly the curves over the edges of direction $\pm e_2$ are fixed, and their image in W is the double curve $\mathsf{D}_0 \ni P, Q$ of Proposition 9.24. In particular X^{μ_p} is a mod-p homology 2-sphere for every prime p, as P. A. Smith's theorem requires of a $\mathbb{Z}/p$-action on a mod-p homology 6-sphere [Bre72, Ch. III]; compare [HKP, Prop. 2.5].

Remark 9.26. Proposition 9.24 realises the first alternative of [HKP, Prop. 2.5]: for $I \cong \mathbb{C}^* \subset \mathrm{Aut}(X)$ with fixed set F, either $F \cong \mathbb{P}^1$ or F is two disjoint points; had $\hat{\delta} \leftrightarrow e_2$ not been an edge direction of the triangulation, the computation above would have produced the second alternative, the two triple points P, Q with opposite tangent weights. Also $h^0(T_X) = 1$ is compatible with [HKP, Cor. 1.3] ($h^0(T_X) \leq 2$ for any complex structure on S^6). The exclusion of a compact $\mathrm{Aut}^0(X)$ in Proposition 9.23 is obtained directly, from the non-constancy of the period matrix (Lemma 9.3).

9.8 Non-Kählerness

Remark 9.27. Since $b_2(X) = 0$ (§7), X carries no Kähler metric, and no Kähler current: by the characterisation of the Fujiki class $\mathcal{C}$ through the existence of a Kähler current (see [Ang18]) such a current would make X bimeromorphic to a compact Kähler manifold, whence the $\partial\bar{\partial}$-lemma and $b_2 \geq h^{1,1} \geq 1$. Also $a(X) = 1 < 3$ shows that X is not Moishezon.

10 Relation to the work of Campana, Demailly and Peternell.

Throughout this section X denotes the compact complex threefold constructed in §§3–6, $f\colon X \to B = \mathbb{P}^1$ the holomorphic map of the Main Theorem, $\Sigma = \{p_0, p_1, p_2\}$ its critical set, $W = f^{-1}(p_0)$, and $S_j = \big(f^{-1}(p_j)\big)_{\mathrm{red}}$, so that $f^*(p_1) = 3S_1$, $f^*(p_2) = 4S_2$; none of this, and in particular not the first proof of $a(X) = 1$ in §§3–9, uses [CDP20] or [CDP98], the one invocation of [CDP20] occurring in Remark 9.9, on which nothing below depends. We use freely:

(X1) $\pi_1(X) = 1$ (Theorem 7.17), $H_*(X;\mathbb{Z}) \cong H_*(S^6;\mathbb{Z})$, hence $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$ and $b_1(X) = b_2(X) = 0$ (Theorem 7.22); X is diffeomorphic to S^6 (Theorem 8.1);

(X2) $a(X) = \operatorname{trdeg}_{\mathbb{C}} \mathcal{M}(X) = 1$, where $\mathcal{M}(\cdot)$ denotes the field of meromorphic functions; the algebraic reduction is f, which is holomorphic with connected fibres, $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$, $R^1 f_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ — locally free and compatible with base change (Proposition 9.13) —, $h^1(X,\mathcal{O}_X) = 1$ and $c_3(X) = e(X) = 2$ (Theorem 9.1);

(X3) W is reduced, irreducible, with normal crossings; its double locus D consists of three rational curves through two triple points, and its normalisation $\eta\colon \widetilde{W} \to W$ is dP_6, with the opposite sides of the hexagon of (-1)-curves identified in pairs (§4);

(X4) every fibre of f over $B^\circ := B \smallsetminus \Sigma$ is a complex 2-torus, and $H^1(F;\mathbb{Q})^{\pi_1(B^\circ)}$ has rank 1 for a general fibre F, spanned by the primitive invariant class γ (§2); here and below we keep the notation of §2: $\Lambda = H_1(F;\mathbb{Z})$ is the period lattice, with frame $(\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta})$, and $V = H^1(F;\mathbb{Z}) = \operatorname{Hom}(\Lambda,\mathbb{Z})$ with frame (γ, u, w, δ). The monodromy-invariant line in $H^1(F;\mathbb{Q}) = V \otimes \mathbb{Q}$ is $\mathbb{Q}\gamma$ (Lemma 2.7), and the co-invariants of H_1 are $\Lambda/\ker\gamma \cong \mathbb{Z}$, detected by γ. Moreover, writing $\varpi\colon \mathcal{H}^\circ \to B^\circ$ for the connected covering of §3 on which the period functions are single valued — the restriction over B° of the orbifold universal cover of $B \smallsetminus \{p_0\}$, *not* the universal cover of B°; its fibres are countable, $\pi_1(B^\circ)$ being countable — the fibres of f over B° are $X_{\varpi(\tilde{q})} = \mathbb{C}^2/\Pi(\tilde{q})\Lambda$ with period matrix

$$\Pi(\tilde{q}) = \begin{pmatrix} 6\mu(\tilde{q}) & \tau(\tilde{q}) & 1 & 0 \\ \beta(\tilde{q}) & \mu(\tilde{q}) & 0 & 1 \end{pmatrix} \qquad (\tilde{q} \in \mathcal{H}^\circ;\ §3),$$

the columns being the images of $\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta}$, and $1, \tau, \mu$ are linearly independent over $\mathbb{C}$ as holomorphic functions on $\mathcal{H}^\circ$ (Lemma 9.3).

By (X1)–(X2), X satisfies the hypotheses of [CDP20, Theorem 2.2, p. 680] while $c_3(X) = 2 > 0$, and X is homeomorphic to S^6 with $a(X) = 1$, so that the conclusion of [CDP20, Corollary 2.3, p. 680] fails for the X constructed here (the existence of such an X being the content of this paper). The purpose of this section is to locate, as precisely as we can, the two places in the printed arguments of [CDP20, CDP98] which do not cover X. They are:

(i) *the reduction "it suffices to show* (11)*"* of [CDP20, §7, p. 692], by which the vanishing $H^0(X_c, \Omega^1_X|X_c \otimes L|X_c) = 0$ — i.e. hypothesis [CDP20, (1), p. 680] of [CDP20, Proposition 2.4, p. 680] — is reduced to the vanishing of $H^0(S, \widetilde{\Omega}^1_S \otimes L|S)$ on the reduction S of the fibre. At the fibre W of our X the conclusion of that reduction is *unconditionally* false: $R^2 f_*(T_X \otimes L) \neq 0$ for *every* $L \in \operatorname{Pic}(X)$ (Theorem 10.5), so hypothesis [CDP20, (1)] holds for no L, [CDP20, Proposition 2.4] is vacuous for X, and the conclusions a) and c) of [CDP20, Theorem 2.2, p. 680] fail for the X constructed here. Statement [CDP20, (11)] itself does hold at W for every twist coming from $\operatorname{Pic}(X)$; what fails at W is the implication from it (§10.4);

(ii) *the monodromy line*: [CDP20, Lemma 4.2, p. 684], which is [CDP98, Lemma 3.2, p. 84], together with the last step of the proof of [CDP98, Theorem 3.1, p. 85] (quoted in, and used for, [CDP20, Corollary 5.2, p. 688] in the torus case). The printed argument passes through the sentence "From (1) we deduce that $H_1(U,\mathbb{Z}) \simeq H_1(W,\mathbb{Z}) \oplus H_1(F,\mathbb{Z})$", which presumes that the monodromy acts trivially on $H_1(F)$; in general a split extension yields only the *co-invariants* (Lemma 10.7), and for X these have rank $t' = 1$ instead of $t = 4$. Here more than a step fails: the *conclusion* of [CDP98, Theorem 3.1] = [CDP20, Corollary 5.2, p. 688] fails for the X constructed here, since $H^1(X, \mathcal{O}_X) \cong \mathbb{C}$ while $H^1(F, \mathcal{O}_F) \cong \mathbb{C}^2$, so the restriction $H^1(X, \mathcal{O}_X) \to H^1(F, \mathcal{O}_F)$ is not surjective; consequently the conclusion of [CDP20, Proposition 5.5, p. 688] ($R^1 f_* \mathcal{O}_X \cong \mathcal{O}(b_1) \oplus \mathcal{O}(b_2)$ with $b_j \geq 0$) fails for the X constructed here, which has $R^1 f_* \mathcal{O}_X \cong \mathcal{O} \oplus \mathcal{O}(-1)$, and the line bundle L with $L|_S$ non-torsion on *every* fibre which [CDP20, Proposition 5.6, p. 689] supplies as input to [CDP20, §7] is not produced for X by the printed argument (§10.5). This last gap, however, is *repairable*: such an L does exist on X (Proposition 10.8), so that the §5 input to [CDP20, §7] can be restored for X by other means, and the monodromy point can be repaired for X.

The scope of the discussion below is the following. What is located at each place is a step of a *printed proof*; where the conclusion of the statement itself does not fail for X, the statement may well survive by other arguments, and this applies in particular to [CDP20, Proposition 5.6] under hypotheses different from those of X, whose *conclusion* is in any case true for X (Proposition 10.8). Where the conclusion of a statement does fail for the X constructed here this is recorded explicitly: so for [CDP20, Theorem 2.2 a), c)] and [CDP20, Corollary 2.3] (through (i)); and at place (ii) it is not only the printed *proofs* but also the *conclusions* of [CDP20, Lemma 4.2]=[CDP98, Lemma 3.2] (the count $r = s - 1 + b_1(X_c)$, which for X would give $r = 6$, whereas the correct count is $r = 3$), of [CDP20, Corollary 5.2]=[CDP98, Theorem 3.1] and of [CDP20, Proposition 5.5] $\big(=$ [CDP98, Remark 3.8(1)]$\big)$ which fail for the X constructed here, these statements presupposing trivial monodromy on H_1 of the fibre; as a located *step*, what is at issue in [CDP98, Lemma 3.2] is the deduction (1)⇒(2), and the corrected count $r = s - 1 + t'$ does hold at X (§10.5). This failure at (ii) is nevertheless *repairable* for the purpose of the proof of [CDP20, Corollary 2.3] at X: the input which [CDP20, §5] is to deliver to [CDP20, §7] — a line bundle non-torsion on every fibre — exists on X by the countability argument of Proposition 10.8, so that the monodromy point can be repaired for X. Place (i) alone already shows that the printed proof of [CDP20, Theorem 2.2] does not apply to X, and the reduction at W cannot be repaired; place (ii) shows in addition that the printed *route* by which [CDP20, §5] delivers its input to [CDP20, §7] breaks down at X, although the input itself is available here by another argument. Nothing below bears on the remaining cases treated in [CDP20, CDP98].

10.1 The statements under discussion

We quote [CDP20] verbatim; numbering inside quotations is that of [CDP20], and page references are to the journal version, Compositio Math. **156** (2020), 679–696.[3] For a coherent sheaf $\mathcal{F}$ on a reduced complex space S we write $\mathcal{T}(\mathcal{F}) \subset \mathcal{F}$ for the subsheaf of sections supported on a nowhere dense closed analytic subset and $\widetilde{\mathcal{F}} := \mathcal{F}/\mathcal{T}(\mathcal{F})$; in particular $\widetilde{\Omega}^1_S = \Omega^1_S/\text{torsion}$.

> **2.2. Theorem** (p. 680). "Let X be a 3-dimensional compact complex manifold with $H^1(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = 0$ and algebraic dimension $a(X) = 1$. Assume that the algebraic reduction $f\colon X \to C$ is holomorphic. Then a) $H^i(X, T_X \otimes M) = 0$ for $i \neq 1$. b) $\chi(X, T_X \otimes M) \leq 0$. c) $c_3(X) \leq 0$."
>
> **2.3. Corollary** (p. 680). "Let X be a 3-dimensional compact complex manifold homeomorphic to S^6. Then $a(X) = 0$."

Here $C \cong \mathbb{P}^1$ and M is a general element of $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X)$, the identification being the one furnished by the exponential sequence. In the proof of [CDP20, Cor. 2.3, p. 680] the case $a(X) = 3$ is excluded at once ("Obviously, $a(X) \neq 3$, otherwise X is Moishezon and therefore $b_2(X) \neq 0$"); $a(X) = 2$, and $a(X) = 1$ with non-holomorphic algebraic reduction, are disposed of by [CDP20, Theorem 2.1, p. 679], the latter written out in detail in [LRS, Theorem 3.1, Corollary 3.2], following [CDP98]. The remaining

[3] The journal text agrees, up to copy-editing, with arXiv:1904.11179v2; we cite the journal as the primary reference and mention the arXiv numbering only where a statement or wording is found there alone.

case — holomorphic algebraic reduction onto a curve, the case of our X — is treated only in [CDP20, Theorem 2.2, p. 680], whose strategy is:

> **2.4. Proposition** (p. 680). "Let X be a 3-dimensional compact complex manifold with $b_2(X) = 0$ and algebraic dimension $a(X) = 1$. Assume that the algebraic reduction $f\colon X \to C$ is holomorphic. If
> $$(1) \qquad R^2 f_*(T_X \otimes L) = 0$$
> for some (or general) $L \in \mathrm{Pic}(X)$, then the assertions of Theorem 2.2 hold. ... The key is to show that the restriction $L|X_c$ of some or the general line bundle L to any fiber is never torsion. Then we compute directly on X_c; here the case when X_c is singular, in particular non-normal, needs special care."

The production of a line bundle L whose restriction to every fibre is non-torsion — the "key" just quoted — is the task of [CDP20, §5], which we examine in §10.5; the verification of [CDP20, (1)] for such an L occupies [CDP20, §7], which opens ([CDP20, p. 692]) by reducing $H^2(X_c, T_X|X_c \otimes L|X_c) = 0$ by Serre duality to $H^0(X_c, \Omega^1_X|X_c \otimes L|X_c) = 0$, and then, with $S = \mathrm{red}\, X_c$, continues:

> "Using the (co)tangent sheaf sequence $0 \to N_S^* \to \Omega^1_X|S \to \Omega^1_S \to 0$ it is immediate, provided $L|S$ is non-torsion, that it suffices to show the statement
> $$(11) \qquad H^0(S, \widetilde{\Omega}^1_S \otimes L|S) = 0,$$
> where $\widetilde{\Omega}^1_S = \Omega^1_S/\text{torsion}$. We first treat smooth fibers $S = X_c$."

(The twist is by L^* in [CDP20, Prop. 2.4, p. 680] and by L in [CDP20, §7, p. 692], the twist by ω_X from Serre duality having been absorbed into L, as we do explicitly in Step 2 below; since L ranges over $\mathrm{Pic}(X)$ this is immaterial.) Statement [CDP20, (11), p. 692] is then established for smooth fibres [CDP20, Prop. 7.1, p. 692], for multiple fibres with smooth reduction [CDP20, Rem. 7.2], for singular reduced fibres [CDP20, Prop. 7.3, pp. 693–695], for multiple fibres with singular reduction [CDP20, Rem. 7.4] and for reducible fibres [CDP20, Prop. 7.5]. We shall see that for our X statement [CDP20, (11)] *does* hold at the fibre W, for every twist coming from $\mathrm{Pic}(X)$, while hypothesis [CDP20, (1)] fails at W for every $L \in \mathrm{Pic}(X)$: the step which does not cover X is the passage, in the sentence just quoted, from the conormal sequence to [CDP20, (11)]. The discussion concerns the fibre W only; the multiple fibres $3S_1$, $4S_2$, and the propositions of [CDP20, §7], which are statements about $\widetilde{\Omega}^1_S \otimes L|_S$, lie outside its scope.

10.2 A lemma on reduced normal crossings fibres

In this subsection N_S^* denotes a conormal sheaf; it is not the nilpotent matrix $N = T_0 - I$ of §2.

Definition 10.1 (Setting)**.** Let $\mathcal{X}$ be a complex manifold of dimension 3, $\mathbb{D} \subset \mathbb{C}$ the unit disc with coordinate t, and $g\colon \mathcal{X} \to \mathbb{D}$ proper, surjective, holomorphic. Put $S := g^{-1}(0)$ and assume: (a) S is reduced, i.e. g vanishes to order one along every irreducible component of S; (b) S has normal crossings: every $p \in S$ admits local coordinates (x, y, z) centred at p in which $g = u \cdot x$, $g = u \cdot xy$ or $g = u \cdot xyz$, with u nowhere vanishing. We write $\mathsf{D} \subset S$ for the singular locus with its reduced structure (locally $\{x = y = 0\}$, respectively the union of the three coordinate axes), $\eta\colon \widetilde{S} \to S$ for the normalisation, and $\widetilde{\mathsf{D}} := \eta^{-1}(\mathsf{D})$ with its reduced structure. Thus S is compact, $\widetilde{S}$ is a smooth compact complex surface (possibly disconnected), η is finite and an isomorphism over $S \setminus \mathsf{D}$, and $\widetilde{\mathsf{D}} \subset \widetilde{S}$ is a divisor. Put $dg := g^*(dt) \in H^0(\mathcal{X}, \Omega^1_{\mathcal{X}})$.

Since S is reduced, its ideal sheaf is locally generated by g, so $N_S^* = \mathcal{O}_{\mathcal{X}}(-S)|_S \cong \mathcal{O}_S$ is freely generated by the class of g, whose image in $\Omega^1_{\mathcal{X}}|_S$ is $dg|_S$. A direct computation in the three local models, with $\Omega^1_{\mathcal{X}}|_S$ free over $\mathcal{O}_S$ on dx, dy, dz, shows that $h \mapsto h\, dg|_S$ is injective (for $g = uxy$ a lift $\tilde{h}$ of a section killing $dg|_S$ satisfies $\tilde{h}x, \tilde{h}y \in (xy)$, so $\tilde{h} \in (x) \cap (y) = (xy)$; for $g = uxyz$ likewise $\tilde{h} \in (xyz)$), whence

$$\Omega^1_S = \big(\Omega^1_{\mathcal{X}}|_S\big)/\mathcal{O}_S \cdot dg|_S. \tag{10.1}$$

Lemma 10.2 (Mayer–Vietoris sequence for a normal crossings surface)**.** In the setting 10.1 the sequence of sheaves on S

$$\begin{aligned} &0 \longrightarrow \mathcal{O}_S \xrightarrow{\alpha} \eta_*\mathcal{O}_{\widetilde{S}} \oplus \mathcal{O}_{\mathsf{D}} \xrightarrow{\beta} \eta_*\mathcal{O}_{\widetilde{\mathsf{D}}} \longrightarrow 0,\\ &\alpha(h) = (\eta^*h,\ h|_{\mathsf{D}}), \qquad \beta(\tilde{h},k) = \tilde{h}|_{\widetilde{\mathsf{D}}} - \eta^*k, \end{aligned} \tag{10.2}$$

is exact.

Proof. Both maps are $\mathcal{O}_S$-linear and the assertion is local. β is surjective because $\mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{\mathsf{D}}}$ is surjective and η is finite; α is injective because S is reduced and η is an isomorphism over a dense open subset. For exactness in the middle only the singular models matter. At a double point $S = \{xy = 0\}$ a section of $\ker\beta$ is $\big((h_1(y,z), h_2(x,z)), k(z)\big)$ with $h_1(0,z) = h_2(0,z) = k(z)$, and $h := h_1(y,z) + h_2(x,z) - h_1(0,z)$ has $\alpha(h) = \big((h_1,h_2),k\big)$; in particular $\mathcal{O}_S \cong \{(h_1,h_2) : h_1(0,z) = h_2(0,z)\}$. At a triple point $S = \{xyz = 0\}$ with branches $S_1 = \{x = 0\}$, $S_2 = \{y = 0\}$, $S_3 = \{z = 0\}$, the condition $\beta = 0$ determines k by the h_i and amounts to $h_1(0,z) = h_2(0,z)$, $h_1(y,0) = h_3(0,y)$, $h_2(x,0) = h_3(x,0)$; then

$$h := h_1(y,z) + h_2(x,z) + h_3(x,y) - h_1(0,z) - h_1(y,0) - h_2(x,0) + h_1(0,0)$$

satisfies $h|_{S_i} = h_i$, hence $h|_{\mathsf{D}} = k$ and $\alpha(h)$ is the given section. □

Tensoring (10.2) with a locally free sheaf E, using finiteness of η and the projection formula, and taking global sections gives

$$H^0(S,E) = \big\{(\tilde{\sigma},\sigma_{\mathsf{D}}) \in H^0(\widetilde{S},\eta^*E) \times H^0(\mathsf{D},E|_{\mathsf{D}})\ :\ \tilde{\sigma}|_{\widetilde{\mathsf{D}}} = \eta^*\sigma_{\mathsf{D}}\big\}. \tag{10.3}$$

The conormal sequence of the divisor S gives, for a line bundle A on S,

$$0 \longrightarrow H^0(S, N_S^* \otimes A) \longrightarrow H^0(S, \Omega^1_{\mathcal{X}}|_S \otimes A) \xrightarrow{\rho} H^0(S, \Omega^1_S \otimes A). \tag{10.4}$$

Lemma 10.3. In the setting 10.1, let A be a holomorphic line bundle on S with $\eta^*A \cong \mathcal{O}_{\widetilde{S}}$, let $e \in H^0(\widetilde{S}, \eta^*A)$ be nowhere vanishing, and put $E := \Omega^1_{\mathcal{X}}|_S \otimes A$. Then:

(a) there is a unique $s \in H^0(S,E)$ with $\eta^*s = \eta^*(dg|_S) \otimes e$ and $s|_{\mathsf{D}} = 0$, and $s \neq 0$; in particular $H^0(S, \Omega^1_{\mathcal{X}}|_S \otimes A) \neq 0$;

(b) the image $\rho(s) \in H^0(S, \Omega^1_S \otimes A)$ in (10.4) is a torsion section supported on D; it vanishes if and only if $A \cong \mathcal{O}_S$ and $e = \eta^*\sigma$ for a trivialising section σ of A. In all cases the image of s in $H^0(S, \widetilde{\Omega}^1_S \otimes A)$ is zero;

(c) if S is connected and $A \not\cong \mathcal{O}_S$, then $H^0(S, N_S^* \otimes A) = H^0(S,A) = 0$; consequently s does not lie in the image of $H^0(S, N_S^* \otimes A) \to H^0(S,E)$.

Proof. Two properties of $dg|_S$, read off the local models: *(i)* $dg|_S$ is nowhere zero on $S \setminus \mathsf{D}$ (for $g = ux$, $dg|_S = u\,dx$); *(ii)* $dg|_{\mathsf{D}} = 0$, since $dg \in (x,y) \cdot \Omega^1_{\mathcal{X}}$ for $g = uxy$ and $dg \in (yz,zx,xy) \cdot \Omega^1_{\mathcal{X}}$ for $g = uxyz$. (Equivalently: dg lies in $\mathfrak{c} \cdot \Omega^1_{\mathcal{X}}|_S$ with $\mathfrak{c}$ the conductor ideal, which for normal crossings is the reduced ideal of D.)

(a) Put $\tilde{\sigma} := \eta^*(dg|_S) \otimes e$, $\sigma_{\mathsf{D}} := 0$; then $\tilde{\sigma}|_{\widetilde{\mathsf{D}}} = 0$ by (ii), so (10.3) yields a unique $s \in H^0(S,E)$, non-zero by (i) and because e is nowhere zero.

(b) On $S \setminus \mathsf{D}$, $s = dg|_S \otimes e$ is a section of $N_S^* \otimes A$ by (10.1); hence $\rho(s)$ vanishes on a dense open set, is supported on D and is torsion. As A is invertible, $\mathcal{T}(\Omega^1_S \otimes A) = \mathcal{T}(\Omega^1_S) \otimes A$, so the image of s in $H^0(S, \widetilde{\Omega}^1_S \otimes A)$ is zero. If $\rho(s) = 0$ then by (10.4) and (10.1) there is $\sigma \in H^0(S,A)$ with $s = dg|_S \otimes \sigma$; pulling back and using that $\eta^*(dg|_S)$ is nowhere zero on the dense open set $\widetilde{S} \setminus \widetilde{\mathsf{D}}$ gives $\eta^*\sigma = e$ there, hence everywhere, so σ trivialises A. The converse is clear.

(c) $N_S^* \cong \mathcal{O}_S$, so $H^0(S, N_S^* \otimes A) = H^0(S,A)$. Let $\sigma \in H^0(S,A)$; then $\eta^*\sigma = \varphi \cdot e$ with φ constant on each connected component of the compact surface $\widetilde{S}$. Let $Z \subset S$ be the union of the irreducible components on which σ vanishes identically. If $Z \neq \emptyset$ and $Z \neq S$, then, S being connected and a finite union of irreducible closed subsets, some component $S' \not\subset Z$ meets Z, say at p. Since η is finite and bimeromorphic, the components of $\widetilde{S}$ correspond bijectively to those of S, the one over S' being $\widetilde{S}' := \overline{\eta^{-1}(S' \setminus \mathsf{D})}$; as $\sigma \not\equiv 0$

on S', the constant $\varphi|_{\widetilde{S}'}$ is non-zero, so σ is nowhere zero on S', contradicting $\sigma(p) = 0$. Hence $\sigma \equiv 0$ or σ is nowhere zero; the latter gives $A \cong \mathcal{O}_S$, excluded. Thus $H^0(S, A) = 0$, and by (a) the non-zero s is not in the image. $\square$

Remark 10.4 (the normal case). Let $S \subset \mathcal{X}$ be a reduced compact fibre of a proper map to a curve, with *S normal*. If $s \in H^0(S, \Omega^1_{\mathcal{X}}|_S \otimes A)$ has torsion image $\rho(s)$, then Ω^1_S is locally free on S_{reg}, so $s|_{S_{\mathrm{reg}}}$ is a section of $N^*_S \otimes A$, which extends across the finite set $S \setminus S_{\mathrm{reg}}$ by the second Riemann extension theorem; so $s \in H^0(S, N^*_S \otimes A)$. Hence if $H^0(S, N^*_S \otimes A) = 0 = H^0(S, \widetilde{\Omega}^1_S \otimes A)$ then $H^0(S, \Omega^1_{\mathcal{X}}|_S \otimes A) = 0$, and the reduction of [CDP20, p. 692] to [CDP20, (11)] is valid. Lemma 10.3 shows that at a non-normal S this extension argument fails.

10.3 Hypothesis (1) of [CDP20, Proposition 2.4, p. 680] fails for X

Theorem 10.5. Let X and $f\colon X \to \mathbb{P}^1$ be as in the Main Theorem. Then:

(a) for every $L \in \mathrm{Pic}(X)$ one has $H^2\big(W, (T_X \otimes L)|_W\big) \neq 0$ and hence $R^2 f_*(T_X \otimes L) \neq 0$; in particular hypothesis [CDP20, (1), p. 680] of [CDP20, Proposition 2.4, p. 680] is satisfied for no $L \in \mathrm{Pic}(X)$;

(b) for every $c \in \mathbb{P}^1$ the restriction $H^1(X, \mathcal{O}_X) \to H^1\big(f^*(c), \mathcal{O}_{f^*(c)}\big)$ to the scheme-theoretic fibre is injective; under the isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X) \cong \mathbb{C}$, $z \mapsto L_z$, of the exponential sequence, the sets $\Lambda_c := \{z \in \mathbb{C} : \ L_z|_{f^*(c)} \cong \mathcal{O}_{f^*(c)}\}$ are countable for all $c \in \mathbb{P}^1$ and discrete for $c \notin \Sigma$. (At $c = p_j$ restriction gives $\Lambda_{p_j} \subset \{z : \ L_z|_{S_j} \cong \mathcal{O}_{S_j}\}$; the opposite inclusion is not claimed.)

(c) for every $M = L_z$ with $z \notin \Lambda_{c_1}$, $c_1 \notin \Sigma$ a fixed point — that is, for general $M \in \mathrm{Pic}(X)$ — one has $H^2(X, T_X \otimes M) \neq 0$.

Proof. Step 1. By (X1), (X2) and the exponential sequence, $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X) \cong \mathbb{C}$ and $c_1(L) = 0$ for all L, so $\mathrm{Pic}(X) = \mathrm{Pic}^0(X)$. As $\widetilde{W} = \mathrm{dP}_6$ by (X3), $H^1(\widetilde{W}, \mathcal{O}) = 0$ and $\mathrm{Pic}(\widetilde{W}) \hookrightarrow H^2(\widetilde{W}, \mathbb{Z})$; since $c_1(\eta^*(L|_W)) = 0$,

$$\eta^*(L|_W) \cong \mathcal{O}_{\widetilde{W}} \qquad \text{for every } L \in \mathrm{Pic}(X). \tag{10.5}$$

Step 2. W is a divisor in X, hence Cohen–Macaulay of pure dimension 2 with invertible dualising sheaf $\omega_W = (\omega_X \otimes \mathcal{O}_X(W))|_W$; since $W = f^*(p_0)$ is reduced, $\mathcal{O}_X(W) = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ restricts trivially to W, so $\omega_W \cong \omega_X|_W$. By Serre–Grothendieck duality on compact complex spaces [RR, RRV], $H^2(W, \mathcal{G})^* \cong H^0(W, \mathcal{G}^\vee \otimes \omega_W)$ for $\mathcal{G}$ locally free; with $\mathcal{G} = (T_X \otimes L)|_W$,

$$H^2\big(W, (T_X \otimes L)|_W\big)^* \cong H^0\big(W, \Omega^1_X|_W \otimes A\big), \qquad A := (L^* \otimes \omega_X)|_W.$$

By (10.5) for $L^* \otimes \omega_X$, $\eta^* A \cong \mathcal{O}_{\widetilde{W}}$; Lemma 10.3(a) applied to $f^{-1}(D) \to D$ for a disc $D \ni p_0$ — legitimate by (X3) — gives $H^0(W, \Omega^1_X|_W \otimes A) \neq 0$, hence $H^2\big(W, (T_X \otimes L)|_W\big) \neq 0$.

Step 3. f is flat (X a connected manifold, $\mathbb{P}^1$ a smooth curve, f surjective) and proper, $T_X \otimes L$ is locally free — hence $T_X \otimes L$ is f-flat, f being flat — and all fibres have dimension 2; cohomology and base change in top degree [BaSt, Ch. III, §3] — the statement invoked in the proof of [CDP98, Lemma 1.5, p. 79][4] — gives that $R^2 f_*(T_X \otimes L) \otimes k(c) \to H^2\big(f^*(c), (T_X \otimes L)|_{f^*(c)}\big)$ is an isomorphism for every c, the fibre being the scheme-theoretic one. With Step 2 this proves (a), $f^*(p_0) = W$ being reduced.

Step 4 (proof of (b)). For every $c \in \mathbb{P}^1$ one has $\mathcal{O}_X\big(-f^*(c)\big) \cong f^*\mathcal{O}_{\mathbb{P}^1}(-1)$, the fibre $f^*(c)$ being the full scheme-theoretic fibre (equal to X_c for $c \notin \{p_1, p_2\}$, and equal to $m_j S_j$ for $c = p_j$). By the projection formula and (X2), $f_* f^* \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_{\mathbb{P}^1}(-1)$ and $R^1 f_* f^* \mathcal{O}_{\mathbb{P}^1}(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, so both outer terms of

$$0 \to H^1\big(\mathbb{P}^1, f_* f^* \mathcal{O}(-1)\big) \to H^1\big(X, f^*\mathcal{O}(-1)\big) \to H^0\big(\mathbb{P}^1, R^1 f_* f^* \mathcal{O}(-1)\big)$$

vanish; hence $H^1(X, f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and, from $0 \to f^*\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_X \to \mathcal{O}_{f^*(c)} \to 0$, the restriction $H^1(X, \mathcal{O}_X) \to H^1\big(f^*(c), \mathcal{O}_{f^*(c)}\big)$ is injective, for every c including $c = p_j$. By functoriality of the exponential sequence on the compact complex space $f^*(c)$, $L_z|_{f^*(c)} \cong \mathcal{O}_{f^*(c)}$ if and only if the image of z in $H^1\big(f^*(c), \mathcal{O}_{f^*(c)}\big)$ lies in the image of the countable group $H^1\big(f^*(c), \mathbb{Z}\big)$ (a finitely generated group, $f^*(c)$

[4] In [CDP98] this reference is cited as [BaSt76]; it is [BaSt] of the present bibliography.

having the homotopy type of a finite complex); so Λ_c, a preimage under a $\mathbb{C}$-linear injection of a countable set, is countable. For $c \notin \Sigma$, $f^*(c) = X_c$ is a 2-torus by (X4), that image is a lattice in $H^1(X_c, \mathcal{O}_{X_c}) \cong \mathbb{C}^2$, and Λ_c is its preimage under an injective map $\mathbb{C} \to \mathbb{C}^2$, hence discrete. Finally, if L_z is trivial on $f^*(p_j) = m_j S_j$ then it is trivial on the closed subspace S_j, which gives the stated inclusion. In particular $\Lambda_{c_1} \neq \mathbb{C}$, so the M of (c) form the complement of a countable set in $\mathrm{Pic}(X) \cong \mathbb{C}$.

Step 5 (proof of (c)). Let $M = L_z$, $z \notin \Lambda_{c_1}$, $c_1 \notin \Sigma$. By (X4) and (X1) each X_c, $c \notin \Sigma$, is a 2-torus with $c_1(M|_{X_c}) = 0$; on a complex torus a topologically trivial line bundle is trivial iff it has a non-zero section, so by Grauert's semicontinuity theorem $\{c \notin \Sigma : M|_{X_c} \cong \mathcal{O}_{X_c}\}$ is a closed analytic subset of the connected curve $\mathbb{P}^1 \setminus \Sigma$, proper by the choice of z, hence discrete. For c outside it, $T_{X_c} \cong \mathcal{O}_{X_c}^{\oplus 2}$ and the normal bundle is trivial, so $(T_X \otimes M)|_{X_c}$ is an extension of $M|_{X_c}$ by $(M|_{X_c})^{\oplus 2}$, and all cohomology of a non-trivial topologically trivial line bundle on a torus vanishes (compare [CDP20, Proposition 7.1(a), p. 692]); hence $H^2\big(X_c, (T_X \otimes M)|_{X_c}\big) = 0$ there.

We claim that the coherent sheaf $R^2 f_*(T_X \otimes M)$ — coherent by Grauert's direct image theorem [BaSt, Chap. III], f being proper — has finite, in particular 0-dimensional, support. Indeed its support is a *closed analytic* subset of $\mathbb{P}^1$, and by Step 3 together with Nakayama's lemma it is exactly $\Xi := \{c \in \mathbb{P}^1 : H^2\big(f^*(c), (T_X \otimes M)|_{f^*(c)}\big) \neq 0\}$, the fibres being the scheme-theoretic ones. By the preceding paragraph $\Xi \setminus \Sigma$ is a discrete subset of $\mathbb{P}^1 \setminus \Sigma$ missing c_1, so $\Xi \neq \mathbb{P}^1$; a proper closed analytic subset of the connected curve $\mathbb{P}^1$ is finite. It is the coherence of $R^2 f_*(T_X \otimes M)$ — i.e. the closedness of Ξ in all of $\mathbb{P}^1$ — which rules out an accumulation of the points of $\Xi \setminus \Sigma$ at Σ: such an accumulation would make the closed analytic set Ξ infinite, hence equal to $\mathbb{P}^1$. By Step 2, $p_0 \in \Xi$ (here $f^*(p_0) = W$), so $R^2 f_*(T_X \otimes M)$ is a non-zero sheaf with finite support and therefore has non-zero global sections. In the Leray spectral sequence $E_2^{p,q} = H^p\big(\mathbb{P}^1, R^q f_*(T_X \otimes M)\big) \Rightarrow H^{p+q}(X, T_X \otimes M)$ one has $E_2^{p,q} = 0$ for $p \geq 2$, so $E_\infty^{0,2} = E_2^{0,2}$ and $H^2(X, T_X \otimes M)$ surjects onto $E_2^{0,2} \neq 0$. ☐

Corollary 10.6. For the threefold X of the Main Theorem: (i) hypothesis [CDP20, (1), p. 680] of [CDP20, Proposition 2.4, p. 680] holds for no $L \in \mathrm{Pic}(X)$, so [CDP20, Proposition 2.4, p. 680] is vacuous for X; (ii) conclusion a) of [CDP20, Theorem 2.2, p. 680] fails for the X constructed here for $i = 2$ and general $M \in \mathrm{Pic}(X)$; conclusion b) fails as well, for *every* $M \in \mathrm{Pic}(X)$, since $c_1(X) = c_2(X) = 0$ (by $H^2(X;\mathbb{Z}) = H^4(X;\mathbb{Z}) = 0$) and $c_1(M) = 0$, so that Riemann–Roch gives $\chi(X, T_X \otimes M) = \frac{1}{2}c_3(X) = 1 > 0$; and conclusion c) fails since $c_3(X) = 2 > 0$; yet X satisfies the hypotheses of that theorem, namely $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$, $a(X) = 1$ and holomorphic algebraic reduction f; (iii) X is diffeomorphic, in particular homeomorphic, to S^6 and has $a(X) = 1$, so that the conclusion of [CDP20, Corollary 2.3, p. 680] fails for the X constructed here.

Proof. (i) and the $i = 2$ part of (ii) are Theorem 10.5(a),(c) with (X1), (X2); for the *b*) part of (ii), $\mathrm{ch}(T_X \otimes M) = 3 + \frac{1}{2}c_3(X)$ and $\mathrm{Td}(X) = 1$ when $c_1(X) = c_2(X) = c_1(M) = 0$, whence $\chi(X, T_X \otimes M) = \frac{1}{2}c_3(X) = 1$ by (X2); the c_3 part of (ii) is (X2); (iii) is (X1), (X2). ☐

10.4 First place: the reduction to [CDP20, (11), p. 692] at the fibre W

Statement [CDP20, (11)] *does* hold at W for every twist coming from $\mathrm{Pic}(X)$. Indeed $\eta_*\Omega^1_{\widetilde{W}}$ is torsion free over $\mathcal{O}_W$ (η finite, $\Omega^1_{\widetilde{W}}$ locally free), so $\Omega^1_W \to \eta_*\Omega^1_{\widetilde{W}}$ kills $\mathcal{T}(\Omega^1_W)$, and the induced map $\widetilde{\Omega}^1_W \to \eta_*\Omega^1_{\widetilde{W}}$, an isomorphism over the dense open set $W \setminus \mathsf{D}$ with $\widetilde{\Omega}^1_W$ torsion free, is injective. Hence for every $B \in \mathrm{Pic}(X)$, by the projection formula and (10.5),

$$H^0\big(W, \widetilde{\Omega}^1_W \otimes B|_W\big) \hookrightarrow H^0\big(\widetilde{W}, \Omega^1_{\widetilde{W}} \otimes \eta^*(B|_W)\big) = H^0\big(\widetilde{W}, \Omega^1_{\widetilde{W}}\big) = 0,$$

as $\widetilde{W} = \mathrm{dP}_6$ is rational; with $B = L^* \otimes \omega_X$ this gives $H^0(W, \widetilde{\Omega}^1_W \otimes A) = 0$ for A as in Step 2. By Theorem 10.5(b), for all z outside the countable set $\bigcup_{n\geq 1} \frac{1}{n}\Lambda_{p_0} \cup \big(w + \bigcup_{n \geq 1} \frac{1}{n}\Lambda_{p_0}\big)$, where $\omega_X = L_w$, both $L_z|_W$ and A are non-torsion — so the proviso "provided $L|S$ is non-torsion" of [CDP20, §7, p. 692] is met at W — and z may in addition be taken outside Λ_{c_1}, so that L_z is simultaneously as in Theorem 10.5(c). For such z, $A \not\cong \mathcal{O}_W$, and Lemma 10.3(c),(a) (W irreducible, hence connected) give

$$H^0(W, N_W^* \otimes A) = 0, \qquad H^0(W, \widetilde{\Omega}_W^1 \otimes A) = 0, \qquad H^0(W, \Omega_X^1|_W \otimes A) \neq 0,$$

exactly the situation of Remark 10.4.

Thus the step of [CDP20, §7, p. 692] which does not cover X is the reduction, in the sentence quoted in §10.1, of the vanishing of $H^0(X_c, \Omega_X^1|X_c \otimes L|X_c)$ to [CDP20, (11)]: the propositions of [CDP20, §7] about singular and reducible fibres are statements about $\widetilde{\Omega}_S^1 \otimes L|_S$, i.e. instances of [CDP20, (11)], and are untouched by what is said here, while by Lemma 10.3 and Remark 10.4 the *reduction* fails at a non-normal normal crossings fibre all of whose pull-backs of line bundles from X to the normalisation are trivial: by (X3) and Step 1, precisely the situation of W. The phrase "here the case when X_c is singular, in particular non-normal, needs special care" of [CDP20, Proposition 2.4, p. 680] identifies the same point.

At this first place, what is located is a step of the printed proof; the obstruction which that step was to overcome is, however, unconditional, and it is the *conclusion* drawn from the step which fails for the X constructed here: by Theorem 10.5(a), $R^2 f_*(T_X \otimes L) \neq 0$ for *every* $L \in \mathrm{Pic}(X)$, so hypothesis [CDP20, (1)] holds for no L whatsoever, [CDP20, Proposition 2.4] is vacuous at X, and the conclusions a) and c) of [CDP20, Theorem 2.2, p. 680] fail for the X constructed here (Corollary 10.6). In particular this place alone already shows that the printed proof of [CDP20, Theorem 2.2, p. 680], and with it that of [CDP20, Corollary 2.3, p. 680], does not cover X, independently of anything in §10.5. No repair of the quoted sentence can restore [CDP20, Theorem 2.2] at X; what an argument covering X would have to control instead is described in §10.6.

A numerical symptom of the second place may be recorded here. For X the singular fibres are W, $3S_1$, $4S_2$, so $s = 3$ and the number of irreducible components of singular fibres is $r = 3$, whereas $b_1(X_c) = 4$ for a smooth fibre; thus [CDP20, Lemma 4.2, p. 684] would give $r = s - 1 + b_1(X_c) = 6$. The discrepancy is explained in §10.5.

10.5 Second place: the monodromy line, [CDP20, Lemma 4.2] = [CDP98, Lemma 3.2], and the last step of [CDP98, Theorem 3.1]

> **4.2. Lemma** ([CDP20, p. 684]). "Let s be the number of singular fibers and r be the numbers of irreducible components of the singular fibers. Then $r = s - 1 + b_1(X_c)$, where X_c is a smooth fiber."

The proof refers to [CDP98, Lemma 3.2, p. 84] (Lemma 5.2 of the arXiv text), of which it is the special case $g = 0$, $b_1 = 0$; the two statements are the same lemma. The set-up of [CDP98] is: $\mathcal{Y}$ a compact complex threefold with $b_2 = 0$, $h\colon \mathcal{Y} \to V'$ holomorphic onto a smooth curve of genus g with connected fibres, $\Sigma' \subset V'$ a finite non-empty set with $A = h^{-1}(\Sigma')$ containing all singular fibres, $B'^{\circ} := V' \setminus \Sigma'$, $U = h^{-1}(B'^{\circ})$, $s = \operatorname{card} \Sigma'$, $t = b_1(F)$ for F a general fibre, $D_1, \ldots, D_r$ the components of A (in [CDP98] Σ' is called Δ and B'° is called W; we keep Σ, W, and also the letters V, Λ of (X4), for the objects of this paper):

> "(1) The natural exact sequence of groups $1 = \pi_2(W) \to \pi_1(F) \to \pi_1(U) \to \pi_1(W) \to 1$ is exact and (non-canonically) split. (2) $b_1(U) = b_1(W) + b_1(F) = 2g + s - 1 + t \leq b_1(X) + s - 1 + t$. (3) $r = s - 1 + t + 2g - b_1(X)$. …(2) From (1) we deduce that $H_1(U, \mathbb{Z}) \simeq H_1(W, \mathbb{Z}) \oplus H_1(F, \mathbb{Z})$."

The last sentence is the step we locate: a split extension of groups does not give a direct sum of abelianisations, but a direct sum with the *co-invariants* of the kernel.

Lemma 10.7. Let $1 \to K \to G \to Q \to 1$ be a split exact sequence of groups. Then $G^{\mathrm{ab}} \cong Q^{\mathrm{ab}} \oplus (K^{\mathrm{ab}})_Q$, the co-invariants being for the conjugation action of Q on K^{ab}.

Proof. Choose a splitting, $G = K \rtimes Q$, and let $K' \subset K$ be generated by $[K, K]$ and all $k^{-1}qkq^{-1}$ ($k \in K$, $q \in Q$). Then K' is normal in G: it is normalised by K since it contains $[K, K]$, and by Q since $q'[K, K]q'^{-1} = [K, K]$ and $q'(k^{-1}qkq^{-1})q'^{-1} = k_1^{-1}q_1k_1q_1^{-1}$ with $k_1 = q'kq'^{-1}$, $q_1 = q'qq'^{-1}$. Moreover $K/K' = (K^{\mathrm{ab}})_Q$ with trivial Q-action, so $G/K' \cong (K^{\mathrm{ab}})_Q \times Q$; as $K' \subset [G, G]$, $G^{\mathrm{ab}} = (G/K')^{\mathrm{ab}} \cong (K^{\mathrm{ab}})_Q \oplus Q^{\mathrm{ab}}$. □

Applied to the split sequence quoted above this gives $H_1(U,\mathbb{Z}) \cong H_1(B'^\circ,\mathbb{Z}) \oplus H_1(F,\mathbb{Z})_{\pi_1(B'^\circ)}$, the action being the monodromy of $U \to B'^\circ$; the co-invariants do not depend on the splitting, two sections differing pointwise by elements of $\pi_1(F)$, whose inner automorphisms act trivially on the abelianisation. Put

$$t' := \operatorname{rk} H_1(F;\mathbb{Q})_{\pi_1(B'^\circ)} = \dim_{\mathbb{Q}} H^1(F;\mathbb{Q})^{\pi_1(B'^\circ)} \ \leq\ b_1(F) = t,$$

the two descriptions agreeing by duality of representations. Thus $H_1(U,\mathbb{Z}) \simeq H_1(B'^\circ,\mathbb{Z}) \oplus H_1(F,\mathbb{Z})$ holds, as far as ranks are concerned, if and only if $\pi_1(B'^\circ)$ acts trivially on $H_1(F;\mathbb{Q})$, and what the argument proves in general is

$$b_1(U) = 2g + s - 1 + t', \qquad \text{hence} \qquad r = b_1(U) - b_1(\mathcal{Y}) = s - 1 + t' + 2g - b_1(\mathcal{Y})$$

($s \geq 1$ is needed for $b_1(B'^\circ) = 2g + s - 1$; in [CDP98] Σ' is indeed required to be "a finite non empty set"). The identity $r = b_1(U) - b_1(\mathcal{Y})$, used in [CDP98, Lemma 3.2(3), p. 84] (Lemma 5.2(3) of the arXiv text), follows from the exact sequence of the pair with rational coefficients

$$H^4(\mathcal{Y};\mathbb{Q}) \to H^4(A;\mathbb{Q}) \to H^5(\mathcal{Y},A;\mathbb{Q}) \to H^5(\mathcal{Y};\mathbb{Q}) \to H^5(A;\mathbb{Q}) = 0,$$

in which $H^4(\mathcal{Y};\mathbb{Q}) \cong H_2(\mathcal{Y};\mathbb{Q}) = 0$ by Poincaré duality and $b_2(\mathcal{Y}) = 0$, $\dim_{\mathbb{Q}} H^4(A;\mathbb{Q}) = r$ (for a compact complex space of pure dimension 2, $H_4(A;\mathbb{Q})$ is freely generated by the $[D_i]$, the singular locus and the pairwise intersections having real dimension ≤ 2), $H^5(\mathcal{Y},A;\mathbb{Q}) \cong H_1(U;\mathbb{Q})$ by Lefschetz duality and $H^5(\mathcal{Y};\mathbb{Q}) \cong H_1(\mathcal{Y};\mathbb{Q})$ by Poincaré duality.

In [CDP20, Lemma 4.2, p. 684], where $g = 0$ and $b_1 = 0$, the count corrected in this way reads $r = s - 1 + t'$. For our X we have $t' = 1$ by (X4): the invariants of $H^1(F;\mathbb{Q}) = V \otimes \mathbb{Q}$ form the line $\mathbb{Q}\gamma$ (Lemma 2.7), equivalently the co-invariants of $H_1(F;\mathbb{Q}) = \Lambda \otimes \mathbb{Q}$ form the line $\Lambda_{\mathbb{Q}}/\ker\gamma \cong \mathbb{Q}$, detected by γ. Hence $r = 3 - 1 + 1 = 3$, in agreement with the three irreducible singular fibres W, S_1, S_2, whereas the printed count $r = 6$ presumes trivial monodromy; so the *conclusion* of [CDP20, Lemma 4.2]=[CDP98, Lemma 3.2], and not only its printed proof, fails for the X constructed here. What is located here is again a step of the printed proof: under hypotheses forcing $t' = t$ the printed conclusion of [CDP20, Lemma 4.2]=[CDP98, Lemma 3.2] stands. That the deduction (1)⇒(2) of [CDP98, Lemma 3.2, p. 84] is not valid in general is seen already in the projective world: let $S_0 = (E \times F_0)/G$ be a bielliptic surface [BHPV, V.5] (called hyperelliptic in [CDP20]), with E, F_0 elliptic curves and G finite acting on E by translations and on F_0 with $F_0/G \cong \mathbb{P}^1$; the first projection gives a holomorphic fibre bundle $S_0 \to E/G$ with fibre F_0 whose monodromy factors through the cyclic group of order $2, 3, 4$ or 6 by which G acts on F_0 modulo translations, an action without non-zero invariants on $H_1(F_0;\mathbb{Q})$. With an elliptic curve E' put $Z := S_0 \times E'$ over E/G; its fibres have $b_1 = 4$ and $t' = 2$, $g = 1$, and removing $s \geq 1$ fibres one finds by Künneth and Lemma 10.7 $b_1(U) = (s+1) + 2 = s + 3$, whereas $2g + s - 1 + b_1(F) = s + 5$. ($Z$ is projective, so it does not satisfy the standing hypotheses of [CDP98, CDP20]; the example bears only on the group-theoretic deduction.)

The same line in [CDP98, Theorem 3.1], and its role at X. The substitution $t \rightsquigarrow t'$ is needed again in the last step of the proof of [CDP98, Theorem 3.1, p. 85] (Theorem 5.1 of the arXiv text), where one reads "By 5.2(2) we conclude that $\operatorname{rk} H^0(W, R^1 f_*(\mathbb{Z})) = t$" — an appeal to the very sentence "$H_1(U,\mathbb{Z}) \simeq H_1(W,\mathbb{Z}) \oplus H_1(F,\mathbb{Z})$" discussed above, and valid as printed only when the monodromy is trivial; in general it yields t'. That theorem asserts the surjectivity of $H^1(\mathcal{Y},\mathcal{O}) \to H^1(F,\mathcal{O}_F)$, and it is the source of [CDP20, Corollary 5.2, p. 688], which reads:

> **5.2. Corollary** (p. 688). "The restriction map $H^1(X,\mathcal{O}_X) \to H^1(X_c,\mathcal{O}_{X_c})$ is surjective for all $c \in C$."

The hypotheses under which this is asserted are the standing hypotheses of [CDP20, §5], which we *paraphrase*: X is a 3-dimensional compact complex manifold with $b_2(X) = 0$ (in the application, $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$) and $a(X) = 1$ whose algebraic reduction $f\colon X \to C$ is holomorphic, the general fibre X_c being a complex 2-torus or a Kodaira surface, and $R^1 f_*\mathcal{O}_X$ is locally free ([CDP20, Proposition 5.1]). The proof of Corollary 5.2 reads "Suppose first that F is a Kodaira surface or a torus. Then the assertion is Theorem 3.1 in [CDP98]; the proof works since we now know that $R^1 f_*(\mathcal{O}_X)$ is locally free". Our X satisfies all of these hypotheses, by (X1), (X2) and (X4). The local freeness of $R^1 f_*\mathcal{O}_X$ demanded here does hold for X — it is local freeness of $R^1 f_*\mathcal{O}_X$ with base change (Proposition 9.13), whose statement gives local freeness together with a splitting; base change then follows by descending induction on q in the cohomology-and-base-change theorem for the flat proper morphism f, all $R^q f_*\mathcal{O}_X$ being locally free [BaSt, Ch. III, §3] — although the printed proof of [CDP20, Proposition 5.1] passes through a step which does not apply to X.[5]

At X, however, more than a step fails: *the conclusion itself fails for the X constructed here*. By (X2), $H^1(X,\mathcal{O}_X)\cong\mathbb{C}$ (Theorem 9.1), while $H^1(F,\mathcal{O}_F)\cong\mathbb{C}^2$ for a smooth fibre F, a complex 2-torus by (X4); hence the restriction

$$H^1(X,\mathcal{O}_X)\longrightarrow H^1(F,\mathcal{O}_F)$$

is not surjective, so the conclusion of [CDP98, Theorem 3.1]=[CDP20, Corollary 5.2, p. 688] fails for the X constructed here. At X the statement therefore cannot be restored by repairing the monodromy line, nor by any other argument. Consistently with this, for X the Leray spectral sequence $E_2^{p,q} = H^p(\mathbb{P}^1, R^q f_*\mathbb{Q}) \Rightarrow H^{p+q}(X;\mathbb{Q})$ forces $d_2\colon E_2^{0,1}\to E_2^{2,0}=H^2(\mathbb{P}^1;\mathbb{Q})\cong\mathbb{Q}$ to be injective ($H^1(X;\mathbb{Q})=0$) and surjective ($H^2(X;\mathbb{Q})=0$), so $H^0(\mathbb{P}^1,R^1 f_*\mathbb{Q})\cong\mathbb{Q}$, in agreement with $t'=1$ and not with $t=4$.

The consequences for [CDP20, §5] are immediate. The statement at issue is:

> **5.5. Proposition** (p. 688). “$R^1 f_*(\mathcal{O}_X)\simeq\mathcal{O}(b_1)\oplus\mathcal{O}(b_2)$ with $b_j\geq 0$.”

(The splitting into line bundles is Grothendieck's on $C\cong\mathbb{P}^1$, $R^1 f_*\mathcal{O}_X$ being locally free of rank 2 under the standing hypotheses; what Proposition 5.5 adds is the positivity $b_j\geq 0$.) Its proof reads “We observe that $R^1 f_*(\mathcal{O}_X)$ is generically spanned by Corollary 5.2, since $H^1(X,\mathcal{O}_X) = H^0(C,R^1 f_*(\mathcal{O}_X))$. Hence $b_j\geq 0$”. The conclusion of [CDP20, Proposition 5.5, p. 688] fails for the X constructed here: $R^1 f_*\mathcal{O}_X\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ by (X2) (§9), a summand of negative degree; so that conclusion (and likewise that of [CDP98, Remark 3.8(1), p. 88], Remark 5.8(1) of the arXiv text) fails for the X constructed here, and is not merely unproved there. Finally, [CDP20, Proposition 5.6, p. 689] uses the positivity of the summands b_j to produce a line bundle $L\in\operatorname{Pic}(X)$ whose restriction $L|_S$ to *every* fibre is non-torsion — the “key” quoted in [CDP20, Proposition 2.4, p. 680], and the input which [CDP20, §5] is to deliver to [CDP20, §7]. That *argument* is unconditionally unavailable at X: the hypothesis $b_j\geq 0$ on which it rests fails at X.

Its *conclusion*, however, is true for X, by a countability argument resting on Proposition 9.13. Theorem 10.5(b) shows that for each *individual* fibre the set of $z\in\mathbb{C}\cong\operatorname{Pic}(X)$ with L_z torsion on that fibre is countable — this is what was used in §10.4 at the single fibre W — but it does not by itself produce a single L non-torsion on *all* fibres simultaneously, the fibres forming an uncountable family and a union of uncountably many countable sets needing not be proper. The periods of f supply the missing uniformity.

Proposition 10.8. There exists $L\in\operatorname{Pic}(X)$ such that $L|_{f^*(c)}$ is non-torsion for every $c\in\mathbb{P}^1$ and, moreover, $L|_{S_j}$ is non-torsion for $j=1,2$; indeed all $z\in\mathbb{C}$ outside the union of $\mathbb{R}$ with a countable set have these properties for $L=L_z$.

Proof. In this proof the letters a,b,c,d are reserved for the coordinates of an integral cohomology class, points of $B^\circ=\mathbb{P}^1\setminus\Sigma$ are denoted q, and points of the covering $\varpi\colon\mathcal{H}^\circ\to B^\circ$ of (X4) are denoted $\tilde{q}$.

Normalisations. By Step 1 of Theorem 10.5, $\operatorname{Pic}(X)=\operatorname{Pic}^0(X)\cong H^1(X,\mathcal{O}_X)\cong\mathbb{C}$, $z\mapsto L_z$. The line bundle $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is a *non-zero* element of this group: by (X2) and the projection formula $h^0\big(X,f^*\mathcal{O}_{\mathbb{P}^1}(1)\big) = h^0\big(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1)\otimes f_*\mathcal{O}_X\big) = h^0\big(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(1)\big) = 2\neq 1 = h^0(X,\mathcal{O}_X)$, since $f_*\mathcal{O}_X=\mathcal{O}_{\mathbb{P}^1}$; hence $f^*\mathcal{O}_{\mathbb{P}^1}(1)\not\cong\mathcal{O}_X$,

[5] In that proof the classical canonical bundle formula for multiple fibres, $\omega_X = f^*(L)\otimes\mathcal{O}_X\big(\sum_i(m_i-1)F_i\big)$ with F_i the reductions of the multiple fibres, is applied. For X the multiplicities are $m_1=3$, $m_2=4$, so that formula would give the exponents $(m_1-1,m_2-1)=(2,3)$ at S_1,S_2; the actual exponents differ. By Theorem 9.1, $K_X\cong f^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_X(2S_2)$, i.e. the exponents are $(0,2)$, not $(m_1-1,m_2-1)=(2,3)$. Such decompositions are not unique, since $\mathcal{O}_X(m_jS_j)=f^*\mathcal{O}_{\mathbb{P}^1}(1)$; but no decomposition with exponents $(2,3)$ exists, for comparing $f^*\mathcal{O}_{\mathbb{P}^1}(-1)\otimes\mathcal{O}_X(2S_2)$ with $f^*\mathcal{O}_{\mathbb{P}^1}(k)\otimes\mathcal{O}_X(2S_1+3S_2)$ would make $\mathcal{O}_X(2S_1+S_2)$ a pull-back from B, whereas it is not: S_1 and S_2 are disjoint, so $\mathcal{O}_X(2S_1+S_2)|_{S_1}=\mathcal{O}_X(S_1)|_{S_1}^2$ is non-trivial, $\mathcal{O}_X(S_1)|_{S_1}$ having order exactly 3, while the restriction of a pull-back from B to a fibre component is trivial. So that printed step does not apply to X. The *statement* of [CDP20, Proposition 5.1] nevertheless holds for X: local freeness is established in Proposition 9.13; base change then follows since all $R^q f_*\mathcal{O}_X$ are locally free [BaSt, Ch. III, §3]. This observation concerns the printed proof only, and it does not affect the proof of [CDP20, Cor. 2.3] at X; the decisive place is (i), §10.4.

and we may and do normalise the coordinate on $\mathrm{Pic}(X) \cong \mathbb{C}$ so that $L_{12} = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. By Proposition 9.13 ($R^1 f_*\mathcal{O}_X$ locally free and compatible with base change) — equivalently by Theorem 10.5(b) — the restriction $\iota_q\colon \mathbb{C} = H^1(X, \mathcal{O}_X) \to H^1(X_q, \mathcal{O}_{X_q})$ is $\mathbb{C}$-linear and injective for every $q \in B^\circ$, and likewise at the remaining fibres. For $q \in B^\circ$ put

$$\Lambda_q := \iota_q^{-1}\big(\mathrm{im}\,[\,H^1(X_q;\mathbb{Z}) \to H^1(X_q, \mathcal{O}_{X_q})\,]\big) \subset \mathbb{C},$$

a discrete subgroup containing 12 (because $f^*\mathcal{O}(1)$ is trivial on every fibre); since Λ_q is a group, $L_z|_{X_q}$ is torsion if and only if $z \in \mathbb{Q} \cdot \Lambda_q$.

If $\mathrm{rk}\,\Lambda_q = 1$ then $\Lambda_q \subset \mathbb{Q} \cdot 12 \subset \mathbb{R}$, so $\mathbb{Q} \cdot \Lambda_q \subset \mathbb{R}$: such q are harmless for $z \notin \mathbb{R}$. We must therefore show that $\{q \in B^\circ : \mathrm{rk}\,\Lambda_q = 2\}$ is countable.

The line $\iota_q(\mathbb{C})$. For a complex torus $H^1(X_q;\mathbb{Z}) \to H^1(X_q, \mathcal{O}_{X_q})$ is injective (its kernel is $H^1(X_q;\mathbb{Z}) \cap H^{1,0} = 0$), so $12 \in \Lambda_q$ determines a unique non-zero $\lambda_0(q) \in H^1(X_q;\mathbb{Z})$ with image $\iota_q(12)$. Uniqueness makes $q \mapsto \lambda_0(q)$ a flat section of the local system $R^1 f_*\mathbb{Z}$ over B°. Indeed, let $D \subset B^\circ$ be a disc over which $R^1 f_*\mathbb{Z}$ is trivialised as $\mathbb{Z}^4_D$ and over which the holomorphic vector bundle $q \mapsto H^1(X_q, \mathcal{O}_{X_q})$ — locally free and compatible with base change by Proposition 9.13 — is trivialised as $D \times \mathbb{C}^2$. In these trivialisations the map $\Psi(q, \lambda) \in \mathbb{C}^2$, the image of $\lambda \in \mathbb{Z}^4$ in $H^1(X_q, \mathcal{O}_{X_q})$, is real-linear in λ and real-analytic in (q, λ) (the periods depending holomorphically on q), and for each q it extends to a real-linear *isomorphism* $\mathbb{R}^4 \to \mathbb{C}^2$, whose inverse is bounded uniformly on compact subsets of D; moreover $q \mapsto \iota_q(12)$ is holomorphic, in particular continuous. Hence if $q_n \to q$ in D then $\Psi\big(q_n, \lambda_0(q_n)\big) = \iota_{q_n}(12)$ stays bounded, so the integral classes $\lambda_0(q_n)$ range over a finite subset of $\mathbb{Z}^4$; along a subsequence they are equal to a fixed λ', and passing to the limit gives $\Psi(q, \lambda') = \iota_q(12)$, whence $\lambda' = \lambda_0(q)$ by uniqueness. So λ_0 is locally constant, i.e. a monodromy-invariant integral class. By (X4) the invariants of $H^1(F;\mathbb{Q})$ form the line $\mathbb{Q}\gamma$, so $\lambda_0 \in \mathbb{Q}\gamma$, and

$$P_q := \iota_q(\mathbb{C}) = \mathbb{C} \cdot \big(\text{image of } \gamma \text{ in } H^1(X_q, \mathcal{O}_{X_q})\big).$$

Since ι_q is injective, $\mathrm{rk}\,\Lambda_q = \mathrm{rk}\{\lambda \in H^1(X_q;\mathbb{Z}) :\ \lambda \mapsto P_q\}$; hence $\mathrm{rk}\,\Lambda_q = 2$ forces the existence of an integral class $\lambda \notin \mathbb{Q}\gamma$ whose image lies on P_q.

The condition in periods. Work on the covering $\varpi\colon \mathcal{H}^\circ \to B^\circ$ and use the frames and period matrix of (X4): over $\tilde{q} \in \mathcal{H}^\circ$ the fibre is $\mathbb{C}^2/\Pi(\tilde{q})\Lambda$, the images of the lattice basis $\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta}$ of $\Lambda = H_1(F;\mathbb{Z})$ being the columns $\pi_{\hat{\gamma}} = (6\mu, \beta)$, $\pi_{\hat{u}} = (\tau, \mu)$, $\pi_{\hat{w}} = (1, 0)$, $\pi_{\hat{\delta}} = (0, 1)$, and $\lambda = a\gamma + bu + cw + d\delta \in V = H^1(F;\mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the functional on Λ with $\lambda(\hat{\gamma}) = a$, $\lambda(\hat{u}) = b$, $\lambda(\hat{w}) = c$, $\lambda(\hat{\delta}) = d$. Under the de Rham identification $H^1(X_q;\mathbb{R}) = \mathrm{Hom}(\Lambda, \mathbb{R})$, every real class is uniquely of the form $\lambda = 2\,\mathrm{Re}(\xi\, dz_1 + \zeta\, dz_2)$ with $(\xi, \zeta) \in \mathbb{C}^2$, its value on a lattice vector $\pi = (\pi^{(1)}, \pi^{(2)})$ being $2\,\mathrm{Re}(\xi\pi^{(1)} + \zeta\pi^{(2)})$; its $(1,0)$-part is $\xi\, dz_1 + \zeta\, dz_2$ and its image in $H^1(X_q, \mathcal{O}_{X_q}) \cong H^{0,1}$ is the conjugate $\bar{\xi}\, d\bar{z}_1 + \bar{\zeta}\, d\bar{z}_2$. Evaluating on the four columns of $\Pi(\tilde{q})$:

$$2\,\mathrm{Re}\,\xi = c, \qquad 2\,\mathrm{Re}\,\zeta = d, \qquad 2\,\mathrm{Re}(\xi\tau + \zeta\mu) = b, \qquad 2\,\mathrm{Re}(6\xi\mu + \zeta\beta) = a. \tag{10.6}$$

For $\lambda = \gamma$ (i.e. $a = 1$, $b = c = d = 0$) the first two equations give $\xi = is$, $\zeta = it$ with $s, t \in \mathbb{R}$, and the third gives $s\ \mathrm{Im}\,\tau + t\ \mathrm{Im}\,\mu = 0$; since $(\mathrm{Im}\,\tau, \mathrm{Im}\,\mu) \neq (0, 0)$ — otherwise $\pi_{\hat{u}} = \tau\pi_{\hat{w}} + \mu\pi_{\hat{\delta}}$ with τ, μ real and the lattice would be degenerate — we get $(\xi, \zeta) = i\varrho\,(\mathrm{Im}\,\mu, -\mathrm{Im}\,\tau)$ with $\varrho \in \mathbb{R}$, and $\varrho \neq 0$ because $\gamma \neq 0$ (the fourth equation of (10.6), the only one in which β occurs, then merely fixes the scale ϱ). As conjugation preserves complex lines and $(\mathrm{Im}\,\mu, -\mathrm{Im}\,\tau)$ is a real vector, the image of a class λ with data (ξ, ζ) lies on P_q if and only if (ξ, ζ) is proportional to $(\mathrm{Im}\,\mu, -\mathrm{Im}\,\tau)$, i.e. if and only if

$$\xi\ \mathrm{Im}\,\tau + \zeta\ \mathrm{Im}\,\mu = 0.$$

Writing $\xi = \frac{c}{2} + is$, $\zeta = \frac{d}{2} + it$ as allowed by the first two equations of (10.6), this single complex equation splits into

$$c\ \mathrm{Im}\,\tau + d\ \mathrm{Im}\,\mu = 0 \qquad \text{and} \qquad s\ \mathrm{Im}\,\tau + t\ \mathrm{Im}\,\mu = 0,$$

while the third equation of (10.6) reads $c\ \mathrm{Re}\,\tau + d\ \mathrm{Re}\,\mu - 2(s\ \mathrm{Im}\,\tau + t\ \mathrm{Im}\,\mu) = b$. The two conditions are therefore together equivalent to $c\ \mathrm{Im}\,\tau + d\ \mathrm{Im}\,\mu = 0$ and $c\ \mathrm{Re}\,\tau + d\ \mathrm{Re}\,\mu - b = 0$, that is, to the single holomorphic equation

$$c\,\tau(\tilde{q}) + d\,\mu(\tilde{q}) - b = 0 \qquad (\tilde{q} \in \mathcal{H}^\circ). \tag{10.7}$$

(Neither the coefficient a nor the period β occurs, as they must not: a and β enter only through the fourth equation of (10.6), which imposes no constraint, and adding multiples of γ does not change the condition. For $\lambda = \gamma$, (10.7) is satisfied identically, as it must be.)

Countability. By Lemma 9.3 the functions $1, \tau, \mu$ are linearly independent over $\mathbb{C}$ on $\mathcal{H}^\circ$; hence (10.7) holds identically only if $b = c = d = 0$, i.e. only for $\lambda \in \mathbb{Z}\gamma$. Consequently, for each of the countably many $\lambda \in V = H^1(F;\mathbb{Z})$ with $(b,c,d) \neq (0,0,0)$ — that is, for each $\lambda \notin \mathbb{Q}\gamma$ — the zero set $Z_\lambda \subset \mathcal{H}^\circ$ of (10.7) is a *proper* analytic subset of the one-dimensional $\mathcal{H}^\circ$, hence discrete and countable. Therefore $\bigcup_\lambda Z_\lambda$ is countable and so is its image $\varpi\big(\bigcup_\lambda Z_\lambda\big) \subset B^\circ$; by the previous paragraph, and since ϖ is surjective, that image contains $\{q \in B^\circ : \operatorname{rk}\Lambda_q = 2\}$. For each such q the bad set $\mathbb{Q} \cdot \Lambda_q$ is countable, so these fibres contribute only countably many bad z in all; the fibres with $\operatorname{rk}\Lambda_q = 1$ contribute only $z \in \mathbb{R}$.

The remaining fibres, and the reductions S_j. At $f^*(p_j) = m_j S_j$ and at W the restrictions $\mathbb{C} \to H^1(f^*(p_j), \mathcal{O})$ and $\mathbb{C} \to H^1(W, \mathcal{O})$ are injective and $\mathbb{C}$-linear by base change (Theorem 10.5(b)), so $\{z : L_z|_{f^*(p_j)} \text{ torsion}\}$ and $\{z : L_z|_W \text{ torsion}\}$ are countable. The same holds on the *reductions* S_j, which are the fibres occurring in §10.4 and §6. Let $\varphi_j\colon \mathbb{C} = H^1(X, \mathcal{O}_X) \to H^1(S_j, \mathcal{O}_{S_j})$ be the restriction, a $\mathbb{C}$-linear map. Since $c_1(L_z) = 0$ in $H^2(X;\mathbb{Z}) = 0$, also $c_1(L_z|_{S_j}) = 0$, so by the exponential sequence on the compact complex space S_j the bundle $L_z|_{S_j}$ is trivial as soon as $\varphi_j(z) = 0$; thus $\varphi_j = 0$ would force $L_z|_{S_j} \cong \mathcal{O}_{S_j}$ for *every* z. This is excluded by the multiplicity of the fibre: $\mathcal{O}_X(S_j) \in \operatorname{Pic}(X) = \operatorname{Pic}^0(X)$, say $\mathcal{O}_X(S_j) = L_{z_j}$, and $\mathcal{O}_X(m_j S_j) = f^*\mathcal{O}_{\mathbb{P}^1}(1) = L_{12}$ gives $z_j = 12/m_j$, while $\mathcal{O}_X(S_j)|_{S_j}$ has order exactly $m_j \geq 2$ in $\operatorname{Pic}(S_j)$ (§6); in particular $L_{12/m_j}|_{S_j} = \mathcal{O}_X(S_j)|_{S_j} \not\cong \mathcal{O}_{S_j}$. Hence φ_j is injective, and since $H^1(S_j;\mathbb{Z})$ is finitely generated (S_j having the homotopy type of a finite complex) the set

$$\{z \in \mathbb{C} :\ L_z|_{S_j} \text{ torsion}\} = \mathbb{Q} \cdot \varphi_j^{-1}\big(\operatorname{im}[\, H^1(S_j;\mathbb{Z}) \to H^1(S_j, \mathcal{O}_{S_j})\,]\big)$$

is countable. Hence every z outside $\mathbb{R}$ together with a countable set gives $L = L_z$ with $L|_{f^*(c)}$ non-torsion for all $c \in \mathbb{P}^1$ and $L|_{S_j}$ non-torsion for $j = 1, 2$. □

Remark 10.9. The proof uses only the rank-one invariance recorded in (X4), which fixes the line P_q, and the linear independence of $1, \tau, \mu$ (Lemma 9.3).

Two conclusions follow. First, at place (ii) it is not only the printed proofs but the *conclusions* of [CDP20, Lemma 4.2]=[CDP98, Lemma 3.2] ($r = s - 1 + b_1(X_c)$ would give 6, our count gives 3), of [CDP20, Corollary 5.2]=[CDP98, Theorem 3.1] and of [CDP20, Proposition 5.5] $\big(=$ [CDP98, Remark 3.8(1)]$\big)$ which fail for the X constructed here, all three presupposing trivial monodromy on H_1 of the fibre; yet this failure is *repairable* for the purpose of the proof of [CDP20, Corollary 2.3] at X, since the line bundle which [CDP20, §5] is to deliver to [CDP20, §7] does exist on X by the countability argument of Proposition 10.8, so that the §5 input can be restored for X even though the printed route to it breaks down. Secondly, at place (i) the situation is different: taking the very L of Proposition 10.8 — and, if one wishes, z also outside the countable sets of §10.4 and Theorem 10.5(c) — the proviso "provided $L|S$ is non-torsion" of [CDP20, §7, p. 692] is met at every fibre, and nevertheless

$$H^0\big(W,\ \Omega^1_X|_W \otimes (L^* \otimes \omega_X)|_W\big) \neq 0$$

by Theorem 10.5, so that $R^2 f_*(T_X \otimes L) \neq 0$ and hypothesis [CDP20, (1)] fails for this L as for every other. Thus the monodromy point can be repaired for X; the reduction at the non-normal fibre cannot. The reduction at [CDP20, p. 692], not the monodromy line, is what [CDP20, Theorem 2.2] would have to do without at X.

10.6 What an argument excluding such an X would have had to control

With hypothesis [CDP20, (1), p. 680] unavailable, the mechanism by which the assertions about χ are obtained — [CDP98, Corollary 1.3] with Serre duality and the Leray spectral sequence — still applies. Recall:

> "1.3 Corollary. Let X be a smooth compact threefold with $b_2(X) = 0$ carrying an effective divisor D. Then $H^0(X, B \otimes L) = 0$ for generic L in $\operatorname{Pic}^0(X)$ and every vector bundle B on X."

(The statement is used, and proved, in the order "for every B, for generic L depending on B"; one generic L serving for every B would be untenable, as $B = L^*$ shows.) The fibres of f are effective divisors and $\mathrm{Pic}(X) = \mathrm{Pic}^0(X)$ by Step 1; hence $H^0(X, T_X \otimes M) = 0$ for general M and, applied to $B = \Omega^1_X \otimes \omega_X$ with Serre duality, $H^3(X, T_X \otimes M) \cong H^0(X, \Omega^1_X \otimes \omega_X \otimes M^*)^* = 0$ for general M. Thus for general M

$$\chi(X, T_X \otimes M) = -h^1(X, T_X \otimes M) + h^2(X, T_X \otimes M),$$

and by Theorem 10.5 and its proof $h^2(X, T_X \otimes M) \geq \dim_{\mathbb{C}}\big(R^2 f_*(T_X \otimes M)\big)_{p_0} > 0$; so $\chi(X, T_X \otimes M) \leq 0$ of [CDP20, Theorem 2.2 b), p. 680] would require a compensating lower bound for $h^1(X, T_X \otimes M)$. For M as in Theorem 10.5(c) all cohomology of $(T_X \otimes M)|_{X_c}$ vanishes for c outside a finite set $\Sigma_1 \supset \Sigma$ — finite because $\{c \notin \Sigma : \ M|_{X_c} \cong \mathcal{O}_{X_c}\} \subset \Xi$ and Ξ is finite, as shown in Step 5 of the proof of Theorem 10.5 — so by Grauert's theorem [BaSt, Chap. III] the sheaves $R^j f_*(T_X \otimes M)$, $j = 0, 1$, vanish over $\mathbb{P}^1 \setminus \Sigma_1$. Hence $f_*(T_X \otimes M)$ is torsion free (a local section over a connected U annihilated by a non-zero function would be a section of the locally free sheaf $T_X \otimes M$ on the connected open set $f^{-1}(U)$ of the connected manifold X vanishing on a non-empty open subset, hence zero) of generic rank 0, so $f_*(T_X \otimes M) = 0$, and $R^1 f_*(T_X \otimes M)$ is a torsion sheaf; the Leray spectral sequence gives $h^1(X, T_X \otimes M) = h^0\big(\mathbb{P}^1, R^1 f_*(T_X \otimes M)\big)$, the total length of that torsion sheaf. So an argument excluding a threefold such as X — or establishing b) and c) of [CDP20, Theorem 2.2, p. 680] for it — would have had to produce a lower bound for the torsion of $R^1 f_*(T_X \otimes M)$ at the degenerate fibres, compensating the torsion of $R^2 f_*(T_X \otimes M)$ at W computed above: information of a different kind — local deformation and obstruction data at the degenerate fibres — from the fibrewise vanishing statements of [CDP20, §7, pp. 692–696], and not obtainable from the §5 input, which even when restored at X (Proposition 10.8) supplies only a line bundle non-torsion on every fibre and no control whatever of torsion at the degenerate ones. For X itself the compensation does not occur: $c_3(X) = 2 > 0$ by (X2).

A Matrices, the fan, and cell structures.

A.1. The matrices and the lattice data

Bases (γ, u, w, δ) of V and $(\hat{\gamma}, \hat{u}, \hat{w}, \hat{\delta})$ of $\Lambda = V^*$ as in §2; columns are images, $\mathfrak{A}(T) = (T^{-1})^t$.

$$T_1 = \begin{pmatrix} 1 & 0 & -6 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 6 & 0 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.1}$$

$$T_0 = (T_1 T_2)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad N = T_0 - I = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.2}$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ -6 & -1 & -1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -6 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 \end{pmatrix}, \tag{A.3}$$

$$M_0 = I - N^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.4}$$

$$\varepsilon = \hat{\gamma} + 2\hat{u} - 4\hat{w}, \qquad \varepsilon' = \hat{\gamma} + 3\hat{u} - 3\hat{w}, \qquad v_1 = \varepsilon, \quad v_2 = -\varepsilon', \tag{A.5}$$

with $A_j = \mathfrak{A}(T_j)$, $M_0 = \mathfrak{A}(T_0)$, and B_0 written in the bases $(\bar{\hat{\gamma}}, \bar{\hat{u}})$ of $\overline{\Lambda}$, $(\hat{w}, \hat{\delta})$ of Λ_{tor}, i.e. $B_0\bar{\hat{\gamma}} = -\hat{\delta}$, $B_0\bar{\hat{u}} = \hat{w}$, the map induced by $M_0 - I = -N^t$. We also use

$$\eta_1 := 2u + w + 3\delta, \qquad \eta_2 := u + w + 2\delta, \qquad \mathfrak{q} := u \wedge w + 6\,\gamma \wedge \delta. \tag{A.6}$$

Lemma A.1. 1. $\det T_j = 1$, $T_1^3 = I$, $T_2^4 = I \neq T_2^2$, $T_0 = I + N$ with $N^2 = 0$, $\operatorname{rk} N = 2$, $N\gamma = Nu = 0$, $Nw = -u$, $N\delta = \gamma$, and $A_1A_2M_0 = I$.

2. $\ker(M_0 - I) = \operatorname{im}(M_0 - I) = \Lambda_{\mathrm{tor}} = \langle \hat{w}, \hat{\delta} \rangle$, $M_0 - I$ has Smith form $\operatorname{diag}(1,1,0,0)$, and $B_0 : \overline{\Lambda} \to \Lambda_{\mathrm{tor}}$ is unimodular.

3. $A_j - I$ has Smith form $\operatorname{diag}(1,1,0,0)$; hence $\Lambda^{A_j} = \ker(A_j - I)$ is saturated of rank 2 and $\Lambda_{A_j} := \Lambda/(A_j - I)\Lambda \cong \mathbb{Z}^2$ is free. Explicitly $\Lambda^{A_1} = \langle \varepsilon, \hat{\delta} \rangle$, $\Lambda^{A_2} = \langle \varepsilon', \hat{\delta} \rangle$, $\Lambda^{\langle A_1, A_2 \rangle} = \mathbb{Z}\hat{\delta}$, and
$$(A_1 - I)\Lambda = \langle -\hat{u} - \hat{w} + \hat{\delta},\ \hat{u} - 2\hat{w} \rangle, \qquad (A_2 - I)\Lambda = \langle -\hat{u} + \hat{w},\ -\hat{u} - \hat{w} + \hat{\delta} \rangle.$$

4. With $\Sigma_j := I + A_j + \cdots + A_j^{m_j - 1}$, an operator on Λ, one has
$$\Sigma_1\hat{\gamma} = 3\varepsilon, \qquad \Sigma_1\hat{u} = 2\hat{\delta}, \qquad \Sigma_1\hat{w} = \hat{\delta}, \qquad \Sigma_1\hat{\delta} = 3\hat{\delta};$$
$$\Sigma_2\hat{\gamma} = 4\varepsilon', \qquad \Sigma_2\hat{u} = \Sigma_2\hat{w} = 2\hat{\delta}, \qquad \Sigma_2\hat{\delta} = 4\hat{\delta},$$
so $\Sigma_1\Lambda = \langle 3\varepsilon, \hat{\delta} \rangle$ and $\Sigma_2\Lambda = \langle 4\varepsilon', 2\hat{\delta} \rangle$, whence $\Lambda^{A_1}/\Sigma_1\Lambda \cong \mathbb{Z}/3$ and $\Lambda^{A_2}/\Sigma_2\Lambda \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$. On V the corresponding operator is the transpose $\Sigma_j^V := I + T_j + \cdots + T_j^{m_j - 1}$, and
$$\Sigma_1^V\gamma = 3\gamma, \qquad \Sigma_1^V u = 6\gamma, \qquad \Sigma_1^V w = -12\gamma, \qquad \Sigma_1^V\delta = \eta_1;$$
$$\Sigma_2^V\gamma = 4\gamma, \qquad \Sigma_2^V u = 12\gamma, \qquad \Sigma_2^V w = -12\gamma, \qquad \Sigma_2^V\delta = 2\eta_2,$$
so $\Sigma_1^V V = \langle 3\gamma, \eta_1 \rangle$ and $\Sigma_2^V V = \langle 4\gamma, 2\eta_2 \rangle$.

5. $[\,T_1 - I \mid T_2 - I\,]$ has Smith form $\operatorname{diag}(1,1,1,0)$; hence $V_G := V/\langle (T_1 - I)V, (T_2 - I)V \rangle \cong \mathbb{Z}$, generated by $\bar{\delta}$, and the smallest $\langle A_1, A_2 \rangle$-stable sublattice of Λ containing Λ_{tor} is $\ker\gamma = \langle \hat{u}, \hat{w}, \hat{\delta} \rangle$.

6. $V^{T_1} = \langle \gamma, \eta_1 \rangle$, $V^{T_2} = \langle \gamma, \eta_2 \rangle$, $V^{T_0} = \ker N = \operatorname{im} N = \langle \gamma, u \rangle$, $V^G = \mathbb{Z}\gamma$; moreover
$$(\Lambda^2 V)^{T_0} = \langle \gamma u, \gamma\delta, uw, \gamma w - u\delta \rangle, \qquad (\Lambda^2 V)^{T_j} = \langle \gamma \wedge \eta_j,\ \mathfrak{q} \rangle,$$
$(\Lambda^2 V)^G = \mathbb{Z}\mathfrak{q}$, and the coinvariants are free of rank one: $(\Lambda^2 V)_G = \mathbb{Z}[\gamma\delta]$ with $[uw] = 6[\gamma\delta]$, $[\mathfrak{q}] = 12[\gamma\delta]$, and $(\Lambda^3 V)_G = \mathbb{Z}[uw\delta]$.

7. On $\Lambda^q\Lambda$ ($q = 0, \ldots, 4$) the kernels of $\Lambda^q M_0 - I$ have ranks $(1,2,4,2,1)$, the images are saturated, and the cokernels are free of the same ranks.

Proof. Direct computation with (A.1)–(A.4); we record the non-mechanical points. (2): $M_0 - I = -N^t$ sends $\hat{\gamma} \mapsto -\hat{\delta}$, $\hat{u} \mapsto \hat{w}$ and kills $\hat{w}, \hat{\delta}$. (3): the columns of $A_1 - I$ are $(0,6,-6,-2)^t, (0,-1,-1,1)^t, (0,1,-2,0)^t, 0$, the first being -2 times the second plus 4 times the third, while two 2×2 minors of the remaining pair are 3 and -1; for $A_2 - I$ they are $(0,0,-6,3)^t, (0,-1,1,0)^t, (0,-1,-1,1)^t, 0$, the first $= -3$ times the second plus 3 times the third, with minors 2 and -1. The first row of A_j is $(1,0,0,0)$, so the first row of $A_j - I$ vanishes. Invariance of $\varepsilon, \varepsilon', \hat{\delta}$ and saturation in rank 2 give Λ^{A_j}. (4): Σ_j kills $\operatorname{im}(A_j - I)$ and Σ_j^V kills $\operatorname{im}(T_j - I)$, and the listed values are the four columns of Σ_j, resp. Σ_j^V, in the two bases; here $\hat{\delta} \in \Sigma_1\Lambda$ comes from $\Sigma_1\hat{w} = \hat{\delta}$ and $2\hat{\delta} \in \Sigma_2\Lambda$ from $\Sigma_2\hat{u} = \Sigma_2\hat{w} = 2\hat{\delta}$ (the values on $\hat{\delta}$ alone give only $m_j\hat{\delta}$), and $\eta_1 = \Sigma_1^V\delta$, $2\eta_2 = \Sigma_2^V\delta$ while $\Sigma_j^V u, \Sigma_j^V w \in \mathbb{Z}\gamma$. (5): the last row of T_j is $(0,0,0,1)$, so the δ-row of $T_j - I$ vanishes; hence $(T_j - I)V \subseteq \langle \gamma, u, w \rangle$, and
$$(T_1 - I)V = \langle 2\gamma + u + w,\ 2\gamma - u \rangle, \qquad (T_2 - I)V = \langle -3\gamma + u,\ u + w \rangle$$
sum to the saturated lattice $\langle \gamma, u, w \rangle$, so $V_G \cong \mathbb{Z}$ generated by $\bar{\delta}$. Dualising, the annihilator $\ker\gamma = \langle \hat{u}, \hat{w}, \hat{\delta} \rangle$ of $V^G = \mathbb{Z}\gamma$ is an $\langle A_1, A_2 \rangle$-stable sublattice of Λ containing Λ_{tor}; duality gives stability, and the minimality of $\ker\gamma$ among stable sublattices containing Λ_{tor} is Lemma 2.7(iv). (6): solve $(\Lambda^2 T_i - I)x = 0$ in the basis $(\gamma u, \gamma w, \gamma\delta, uw, u\delta, w\delta)$; the images of $\Lambda^2 T_1 - I$, $\Lambda^2 T_2 - I$ span the saturated rank-5 lattice

$\langle \gamma u, \gamma w, u\delta, w\delta, uw - 6\gamma\delta \rangle$. For $\Lambda^3 V$ use the G-equivariant isomorphism $\Lambda^3 V \cong \Lambda \otimes \Lambda^4 V \cong \Lambda$ (contraction against the invariant generator $\gamma \wedge u \wedge w \wedge \delta$ of $\Lambda^4 V$, invariant since $\det T_j = 1$), under which $\hat{\gamma} \mapsto u \wedge w \wedge \delta$; since by (3) $(A_1 - I)\Lambda + (A_2 - I)\Lambda = \langle \hat{u}, \hat{w}, \hat{\delta} \rangle = \ker \gamma$ is saturated of rank 3, we get $(\Lambda^3 V)_G \cong \Lambda_G = \Lambda / \ker \gamma \cong \mathbb{Z}$, generated by the class dual to γ, i.e. by $[uw\delta]$. (7): for $q = 3$ use $\Lambda^3 \Lambda \cong \Lambda^* \otimes \Lambda^4 \Lambda$ together with $\det M_0 = 1$ and $M_0^t = T_0^{-1}$: under this isomorphism $\Lambda^3 M_0$ corresponds to $(M_0^t)^{-1} = T_0$ acting on $\Lambda^* = V$, so $\Lambda^3 M_0 - I$ becomes $T_0 - I = N$, whose kernel and image are both the saturated rank-2 lattice $\langle \gamma, u \rangle$ by (1); the case $q = 1$ is (2); and for $q = 2$ the image is spanned by two vectors whose 2×2 minors have gcd 1. □

A.2. The fan $\mathfrak{F}$ and the surfaces D_v

Let $N' = \mathbb{Z}^3$ with height y_3 and let $\mathcal{T}_{A_2}$ be the triangulation of $\mathbb{R}^2$ with vertices $\mathbb{Z}^2$ and triangles $\{v, v + e_1, v + e_2\}$, $\{v + e_1, v + e_2, v + e_1 + e_2\}$, with edge directions $\pm e_1, \pm e_2, \pm(e_1 - e_2)$; $\mathfrak{F}$ consists of 0 and the cones over the faces of $\mathcal{T}_{A_2} \times \{1\}$.

Lemma A.2. 1. Every cone of $\mathfrak{F}$ is unimodular and any two meet in a common face; hence $Y = Y_{\mathfrak{F}}$ is a smooth separated toric complex manifold with dense torus $T_{N'}$, and $t = \chi^{(0,0,1)}$ is a global holomorphic function with $t^{-1}(\mathbb{C}^*) = T_{N'}$ and $t^{-1}(0) = \bigcup_{v \in \mathbb{Z}^2} D_v$, D_v the prime divisor of the ray through $(v, 1)$. In the chart $U_\sigma \cong \mathbb{C}^3$ of a triangle σ, with coordinates z_1, z_2, z_3 dual to its vertex rays, $t = z_1 z_2 z_3$.

2. The star of a vertex v consists of the six triangles $\{v, v + a, v + b\}$ with (a, b) a consecutive pair of the cyclic list $e_1, e_2, e_2 - e_1, -e_1, -e_2, e_1 - e_2$. Under $N' \to N'_v := N'/\mathbb{Z}(v, 1)$, $(y, y_3) \mapsto y - y_3 v$, these six neighbours map to that list, so the star fan of $\mathbb{R}_{\geq 0}(v, 1)$ is the hexagonal fan of dP_6 and $D_v \cong \mathrm{dP}_6$ for every v.

3. $D_v \cap D_{v'} \neq \emptyset$ iff $\{v, v'\}$ is an edge of $\mathcal{T}_{A_2}$, and then $D_v \cap D_{v'} \cong \mathbb{P}^1$ is the invariant curve of the ray $v' - v$ in the star fan of v, one of the six (-1)-curves of the hexagon of D_v. Triple intersections are single reduced points, one per triangle.

4. For $\bar{\lambda} \in \overline{\Lambda}$, $\varphi_{\bar{\lambda}}(y, y_3) = (y + y_3 B_0 \bar{\lambda}, y_3)$ lies in $GL(N')$, preserves y_3 and acts on height one by $v \mapsto v + B_0 \bar{\lambda}$; as $\mathcal{T}_{A_2}$ is $\mathbb{Z}^2$-invariant, $\varphi_{\bar{\lambda}}(\mathfrak{F}) = \mathfrak{F}$ and $\Phi_{\bar{\lambda}}(D_v) = D_{v + B_0 \bar{\lambda}}$, and since $B_0 \in GL_2(\mathbb{Z})$ the action of $\overline{\Lambda}$ on $\{D_v\}$ is simply transitive.

5. The group actually acting on Y_ε is $\{\Psi_{\bar{\lambda}}\}$, with $\Psi_{\bar{\lambda}} = \tau_{\bar{\lambda}} \circ \Phi_{\bar{\lambda}}$, where the translation part $\tau_{\bar{\lambda}}$ is translation, at a point p, by the t-dependent torus element $(c_{\bar{\lambda}}(t(p)), 1) \in T_{N'}$; in particular $\tau_{\bar{\lambda}}$ preserves every $T_{N'}$-orbit of Y (in particular every D_v, every $D_v \cap D_{v'}$ and every triple point); hence the orbit structure of $\{\Psi_{\bar{\lambda}}\}$ on the components and strata of $t^{-1}(0)$ coincides with that of $\{\Phi_{\bar{\lambda}}\}$ described in (4), and the conclusions of (4) may be applied verbatim to the threefold quotient $N_0 = Y_\varepsilon / \{\Psi_{\bar{\lambda}}\}$ and to its central fibre $W = t^{-1}(0) / \{\Psi_{\bar{\lambda}}\}$ (see Proposition 4.6). Consequently W is irreducible, $\nu : \mathrm{dP}_6 \cong D_0 \hookrightarrow t^{-1}(0) \to W$ is the normalisation, the three pairs of opposite sides of the hexagon (the pairs $\{0, a\}, \{0, -a\}$, $a = e_1, e_2, e_1 - e_2$) are glued, giving $\mathsf{D} = \bar{C}_1 \cup \bar{C}_2 \cup \bar{C}_3$ with $\bar{C}_i \cong \mathbb{P}^1$, and the six torus-fixed points of D_0 fall into two translation classes, the triple points P, Q. Hence $e(W) = 2$.

6. For $0 < \rho \leq \eta_0$, the tube $\{|t| < \rho\}$ deformation retracts onto $t^{-1}(0) = \bigcup_v D_v$; in particular the inclusion $t^{-1}(0) \hookrightarrow \{|t| < \rho\}$ is a homotopy equivalence, so $\pi_1(\{|t| < \rho\}) \cong \pi_1(t^{-1}(0))$ and $H_*(\{|t| < \rho\}; \mathbb{Z}) \cong H_*(t^{-1}(0); \mathbb{Z})$.

Proof. (1) For a triangle v, v', v'' of $\mathcal{T}_{A_2}$ one has $\det(v' - v, v'' - v) = \pm 1$, hence

$$\det\big((v, 1), (v', 1), (v'', 1)\big) = \pm 1;$$

the face condition is exactly that $\mathcal{T}_{A_2}$ is a triangulation. Smoothness, separatedness and the orbit–cone correspondence are conewise conditions, hence remain valid for infinite fans [Ful93, Oda88, KKMS, AMRT, Mum72]. The character $m = (0,0,1) \in M' = (N')^*$ satisfies $\langle m, (v,1)\rangle = 1$ for *every* primitive ray generator $(v,1)$ of $\mathfrak{F}$; in particular $\langle m, \cdot\rangle \geq 0$ on each cone, so $t = \chi^m$ is a global holomorphic function, vanishing to order one along each D_v. In the chart U_σ of a triangle the three primitive generators form a basis of N' and m takes the value 1 on each of them, so $t = z_1 z_2 z_3$ there. (2) Inspection plus the standard fan of dP_6 [Ful93, Oda88]. (3), (4) Orbit–cone correspondence and direct computation. (5) Since $\tau_{\bar\lambda}$ is translation by the torus element $(c_{\bar\lambda}(t(p)), 1) \in T_{N'}$, it preserves each torus orbit, so the Ψ-orbits of strata are the Φ-orbits of (4); two star-edges $\{0,a\}, \{0,a'\}$ are $\mathbb{Z}^2$-equivalent iff $a' = -a$, and the six star-triangles form two orbits; $e(W)$ is computed orbitwise on $W = (\mathbb{C}^*)^2 \sqcup (\text{three } \mathbb{C}^*) \sqcup \{P, Q\}$.

(6) Proposition 7.2 provides a deformation retraction of the *closed* tube $\{|t| \leq \eta\}$ onto $t^{-1}(0)$ for every radius $\eta \leq \eta_0$; we therefore first retract the open tube onto a closed one.

Step 1 (open tube → closed tube). Let $n_0 = (0,0,1) \in N'$ and let $\lambda_{n_0} : \mathbb{C}^* \to T_{N'}$ be the corresponding one-parameter subgroup. It acts algebraically on the whole of Y (the torus acts on any toric variety) and preserves every $T_{N'}$-orbit closure, in particular the invariant divisor $t^{-1}(0) = \bigcup_v D_v$. On the dense torus, $t(\lambda_{n_0}(s)\cdot y) = \chi^m(\lambda_{n_0}(s))\, t(y) = s^{\langle m, n_0\rangle} t(y) = s\, t(y)$ with $m = (0,0,1)$ as in (1), hence by continuity

$$t(\lambda_{n_0}(s)\cdot y) = s\, t(y) \qquad (y \in Y,\ s \in \mathbb{C}^*).$$

Put $s(y) := \min\{1,\ \rho/(2|t(y)|)\} \in (0,1]$ for $|t(y)| < \rho$ (with $s \equiv 1$ on $\{|t| \leq \rho/2\}$); s is continuous, and

$$H_\tau(y) := \lambda_{n_0}\big(s(y)^\tau\big)\cdot y, \qquad \tau \in [0,1],$$

is continuous in (y,τ) and satisfies $|t(H_\tau(y))| = s(y)^\tau |t(y)| \leq |t(y)| < \rho$, so $H_\tau(\{|t| < \rho\}) \subseteq \{|t| < \rho\}$; moreover $H_0 = \mathrm{id}$, $|t(H_1(y))| = \min\{|t(y)|, \rho/2\}$, and H_τ fixes $\{|t| \leq \rho/2\}$ (hence $t^{-1}(0)$) pointwise. Thus H is a strong deformation retraction of $\{|t| < \rho\}$ onto the closed tube $\{|t| \leq \rho/2\}$.

Step 2 (closed tube → divisor). Since $\rho/2 \leq \eta_0$, Proposition 7.2, applied to the reduced normal-crossing divisor $t^{-1}(0)$ and to the radius $\rho/2$, gives a deformation retraction of $\{|t| \leq \rho/2\}$ onto $t^{-1}(0)$; this is the retraction constructed in §7.2 (polar decomposition, honeycomb subdivision, and the lifted deformation retraction). Composing the two retractions proves (6); the statements on π_1 and H_* follow. (Alternatively, for those two statements one may avoid Step 1 altogether: $\{|t| < \rho\} = \bigcup_n \{|t| \leq \rho_n\}$ for any $\rho_n \uparrow \rho$ with $\rho_n \leq \eta_0$, each inclusion $t^{-1}(0) \hookrightarrow \{|t| \leq \rho_n\}$ is a homotopy equivalence by Proposition 7.2, and loops and singular cycles have compact image, so the colimit gives the isomorphisms for π_1 and H_*.) Finally, the nerve of the covering $\{D_v\}$ is $\mathcal{T}_{A_2}$, but its members meet in $\mathbb{P}^1$'s rather than in contractible sets, so the nerve theorem is not available for this covering. □

A.3. Cell structures and homology used in §7

Lemma A.3 (Homology of W). $H_*(W;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^4, \mathbb{Z}^2, \mathbb{Z})$ for $* = 0, \dots, 4$, torsion free; $e(W) = 2$. Moreover $H_2(W;\mathbb{Z})$ is generated by $\bar C_1, \bar C_2, \bar C_3$ and by the class $\mathbb{T}_B$ of §7; for the identification of this last generator see Proposition 7.11.

Proof. W is the pushout of $\mathsf{D} \leftarrow \widetilde{\mathsf{D}} \to \widetilde{W} = \mathrm{dP}_6$ along closed cofibrations, whence the Mayer–Vietoris sequence [Hat02]

$$\cdots \to H^k(W) \to H^k(\widetilde{W}) \oplus H^k(\mathsf{D}) \to H^k(\widetilde{\mathsf{D}}) \to H^{k+1}(W) \to \cdots,$$

with $H^*(\mathsf{D}) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, 0, 0)$, $H^*(\widetilde{\mathsf{D}}) = (\mathbb{Z}, \mathbb{Z}, \mathbb{Z}^6, 0, 0)$, $H^*(\mathrm{dP}_6) = (\mathbb{Z}, 0, \mathbb{Z}^4, 0, \mathbb{Z})$. The hexagon cycle dies in $H_1(\mathsf{D})$ (it traverses each edge of the underlying theta-graph twice with opposite signs), so $H^1(\mathsf{D}) \to H^1(\widetilde{\mathsf{D}})$ vanishes and $H^1(W) \cong \mathbb{Z}^2$. In degree 2 order the hexagon as $C_1 = E_1$, $C_2 = H - E_1 - E_2$, $C_3 = E_2$, $C_4 = H - E_2 - E_3$, $C_5 = E_3$, $C_6 = H - E_3 - E_1$, opposite pairs $\{C_i, C_{i+3}\}$; the map to $\mathbb{Z}^6$ is

$$\big(D, \textstyle\sum_i \lambda_i \bar C_i^\vee\big) \longmapsto \big((D\cdot C_k)_k - \textstyle\sum_i \lambda_i(\delta_{i,k} + \delta_{i+3,k})_k\big).$$

Its first summand is injective (C_k span $H^2(\mathrm{dP}_6)$, the form being unimodular) and the two images overlap exactly in the $D = aH - \sum b_i E_i$ with $b_1 + b_2 + b_3 = a$, a rank-3 condition; so the map has rank $4 + 3 - 3 = 4$ and its kernel is free of rank 3. Since $H^1(\widetilde{W}) = H^1(\mathrm{dP}_6) = 0$ and, as just noted, $H^1(\mathsf{D}) \to H^1(\widetilde{\mathsf{D}})$ vanishes, the connecting map $\partial : H^1(\widetilde{\mathsf{D}}) = \mathbb{Z} \to H^2(W)$ is injective; its image is the extra $\mathbb{Z}$ summand, so that $H^2(W) \cong \mathbb{Z} \oplus \mathbb{Z}^3$, the $\mathbb{Z}^3$ being the above kernel. The cokernel in degree 2 is $\mathbb{Z}^6/\langle e_i + e_{i+3}, (0,1,0,1,0,1)\rangle \cong$

$\mathbb{Z}^2$, whence $H^3(W) \cong \mathbb{Z}^2$, $H^4(W) = \mathbb{Z}$. Universal coefficients gives the table, and $e(W) = 2$ as in Lemma A.2(5). $\square$

Lemma A.4 (Homology of S_j and the specialisation maps)**.** For $j = 1,2$ put $A_0^{(j)} := \mathbb{C}^2/L(z_j)$ and $S_j = A_0^{(j)}/\langle x \mapsto A_j x + v_j/m_j \rangle$. Then $H_*(S_j;\mathbb{Z}) = (\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}^2,\mathbb{Z}^2,\mathbb{Z})$, torsion free, $e(S_j) = 0$; and for the degree-m_j unramified cover $\pi_j : A_0^{(j)} \to S_j$ the maps $\pi_j^* : H^q(S_j;\mathbb{Z}) \to (\Lambda^q V)^{T_j}$ are injective of index $(1,3,1,1,3)$ for $j = 1$ and $(1,4,2,2,4)$ for $j = 2$ ($q = 0,\dots,4$), with

$$\pi_1^* H^1(S_1) = \langle 3\gamma, \eta_1 \rangle, \qquad \pi_2^* H^1(S_2) = \langle 4\gamma, \eta_2 \rangle,$$

$\pi_1^* H^2(S_1) = (\Lambda^2 V)^{T_1}$, and $\pi_2^* H^2(S_2) = \{a\,\gamma \wedge \eta_2 + b\,\mathfrak{q} :\ a \equiv b\ (2)\}$, of index 2 with cokernel generated by $[\mathfrak{q}]$.

Proof. Write $\Gamma_j := \pi_1(S_j) = \langle \Lambda, \tilde{g} \mid \tilde{g}\lambda\tilde{g}^{-1} = A_j\lambda,\ \tilde{g}^{m_j} = t_{v_j} \rangle$, the last relation because $\Sigma_j(v_j/m_j) = v_j$ for $v_j \in \Lambda^{A_j}$. Abelianising, $H_1(S_j) = (\Lambda_{A_j} \oplus \mathbb{Z}\tilde{g})/\mathbb{Z}([v_j], -m_j)$ with $\Lambda_{A_j} \cong \mathbb{Z}^2$ free by Lemma A.1(3). The vector $([v_j], -m_j)$ is primitive: clear for $p \nmid m_j$, while for $p \mid m_j$ one has $[v_j] \neq 0$ in Λ_{A_j}/p since $(A_j - I)\Lambda \subset \ker\gamma$ and $\gamma(v_1) = 1$, $\gamma(v_2) = -1$. So $b_1 = 2$; as S_j is closed oriented with $e(S_j) = e(A_0^{(j)})/m_j = 0$, $b_2 = 2$, and Poincaré duality with universal coefficients gives the table; in particular $H^q(S_j;\mathbb{Z})$ is torsion free for all q.

Injectivity of π_j^*. For the m_j-fold covering π_j one has $\pi_{j*}\pi_j^* = m_j \cdot \mathrm{id}$ on $H^q(S_j;\mathbb{Z})$; since $H^q(S_j;\mathbb{Z})$ is torsion free, multiplication by m_j is injective, hence so is π_j^*. (The edge homomorphism of the spectral sequence below is in general only surjective onto $E_\infty^{0,q}$, so it yields no injectivity statement.)

The image. $A_0^{(j)}$ is aspherical and $G_j := \Gamma_j/\Lambda \cong \mathbb{Z}/m_j$ acts freely, so $S_j = K(\Gamma_j, 1)$ and the Lyndon–Hochschild–Serre spectral sequence of $1 \to \Lambda \to \Gamma_j \to G_j \to 1$ reads $E_2^{p,q} = H^p(G_j;\Lambda^q V) \Rightarrow H^{p+q}(S_j;\mathbb{Z})$, the composite of π_j^* with the inclusion of $E_\infty^{0,q}$ being the edge map; thus

$$\pi_j^* H^q(S_j;\mathbb{Z}) = E_\infty^{0,q} \subseteq E_2^{0,q} = (\Lambda^q V)^{T_j}.$$

By $\tilde{g}^{m_j} = t_{v_j}$ the extension class in $H^2(G_j;\Lambda) = \Lambda^{A_j}/\Sigma_j\Lambda$ is $[v_j]$. For $q = 1$ the five-term exact sequence gives $E_\infty^{0,1} = \ker\big(d_2 : V^{T_j} \to H^2(G_j;\mathbb{Z}) = \mathbb{Z}/m_j\big)$ with $d_2\xi = \xi(v_j)$ (evaluation of the extension class). For $q = 2$ only d_2 can leave $E^{0,2}$, since $d_3 : E_3^{0,2} \to E_3^{3,0} = H^3(G_j;\mathbb{Z}) = 0$; so $E_\infty^{0,2} = \ker\big(d_2 : (\Lambda^2 V)^{T_j} \to H^2(G_j;V)\big)$. We use d_2 only on *decomposable* invariant classes, where it is given by the multiplicative structure and the derivation property: for $\xi, \xi' \in V^{T_j} = E_2^{0,1}$,

$$d_2(\xi \wedge \xi') = (d_2\xi)\,[\xi'] - (d_2\xi')\,[\xi] = \xi(v_j)[\xi'] - \xi'(v_j)[\xi] = [\iota_{v_j}(\xi \wedge \xi')] \in H^2(G_j;V) = V^{T_j}/\Sigma_j^V V,$$

i.e. by contraction with v_j; for the general description of $d_2^{0,q}$ for extensions with free abelian kernel see Charlap–Vasquez [CV66, CV69] and Huebschmann [Hue81].

$q = 1$: $H^2(G_j;\mathbb{Z}) = \mathbb{Z}/m_j$ and $d_2(\xi) = \xi(v_j)$, so $\pi_j^* H^1(S_j) = \{\xi \in V^{T_j} : \xi(v_j/m_j) \in \mathbb{Z}\}$. Since $\gamma(v_1) = 1$, $\gamma(v_2) = -1$ and $\eta_1(v_1) = \eta_2(v_2) = 0$, this is $\langle 3\gamma, \eta_1 \rangle$, resp. $\langle 4\gamma, \eta_2 \rangle$, of index 3, resp. 4.

$q = 2$: we use the cup-product form. With the orientation class $\gamma \wedge u \wedge w \wedge \delta$ of $A_0^{(j)}$ and (A.6) one computes in $\Lambda^4 V$

$$(\gamma \wedge \eta_j)^2 = 0, \qquad (\gamma \wedge \eta_1) \wedge \mathfrak{q} = 3, \qquad (\gamma \wedge \eta_2) \wedge \mathfrak{q} = 2, \qquad \mathfrak{q} \wedge \mathfrak{q} = 12.$$

By the projection formula, $\langle \pi_j^*\alpha \cup \pi_j^*\beta, [A_0^{(j)}] \rangle = m_j \langle \alpha \cup \beta, [S_j] \rangle$, so the Gram matrix of the lattice $L_j := \pi_j^* H^2(S_j;\mathbb{Z})$, in a basis coming from a basis of the unimodular lattice $H^2(S_j;\mathbb{Z})$, equals m_j times a unimodular matrix; hence $\det L_j = \pm m_j^2$. On the other hand $\det(\Lambda^2 V)^{T_1} = \det\left(\begin{smallmatrix} 0 & 3 \\ 3 & 12 \end{smallmatrix}\right) = -9$ and $\det(\Lambda^2 V)^{T_2} = \det\left(\begin{smallmatrix} 0 & 2 \\ 2 & 12 \end{smallmatrix}\right) = -4$, so if $k_j := [(\Lambda^2 V)^{T_j} : L_j]$ then $9k_1^2 = 9$ and $4k_2^2 = 16$, i.e.

$$k_1 = 1, \qquad k_2 = 2,$$

in agreement with Proposition 7.14, which gives these indices. For $j = 1$ this already yields $\pi_1^* H^2(S_1) = (\Lambda^2 V)^{T_1}$.

For $j = 2$ we must single out one of the three subgroups of index 2 in $\langle \gamma \wedge \eta_2, \mathfrak{q} \rangle$. First, by the derivation computation above and $\gamma(v_2) = -1$, $\eta_2(v_2) = 0$,

$$d_2(\gamma \wedge \eta_2) = -[\eta_2] \in V^{T_2}/\Sigma_2^V V = \langle \gamma, \eta_2 \rangle / \langle 4\gamma, 2\eta_2 \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$$

(Lemma A.1(4)), and $[\eta_2] \neq 0$ there; hence $\gamma \wedge \eta_2 \notin L_2$, which excludes the subgroup $\langle \gamma \wedge \eta_2, 2\mathfrak{q} \rangle$. The two remaining candidates are $L_2 = \langle 2\gamma \wedge \eta_2,\ a\,\gamma \wedge \eta_2 + \mathfrak{q} \rangle$ with $a \in \{0, 1\}$. Writing e, f for the corresponding basis of $H^2(S_2; \mathbb{Z})$, the above products give

$$\langle e, e \rangle = \tfrac{1}{4}(2\gamma \wedge \eta_2)^2 = 0, \qquad \langle e, f \rangle = \tfrac{1}{4} \cdot 2 \cdot 2 = \pm 1, \qquad \langle f, f \rangle = \tfrac{1}{4}\big(4a + 12\big) = a + 3.$$

Now S_2 is a compact complex surface and π_2 is étale, so $\pi_2^* K_{S_2} = K_{A_0^{(2)}} = \mathcal{O}$ and $K_{S_2}^{\otimes m_2}$ is trivial; thus $c_1(S_2)$ is torsion in $H^2(S_2; \mathbb{Z})$, which is torsion free, so $c_1(S_2) = 0$ and $w_2(S_2) = c_1(S_2) \bmod 2 = 0$. Hence S_2 is spin and its intersection form is even, forcing $a + 3$ to be even, i.e. $a = 1$:

$$\pi_2^* H^2(S_2) = \langle 2\gamma \wedge \eta_2,\ \gamma \wedge \eta_2 + \mathfrak{q} \rangle = \{a\,\gamma \wedge \eta_2 + b\,\mathfrak{q} :\ a \equiv b\ (2)\},$$

of index 2, the cokernel being generated by the class of $\mathfrak{q}$. (Consistently, for $j = 1$ the form $\frac{1}{3}\left(\begin{smallmatrix} 0 & 3 \\ 3 & 12 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 4 \end{smallmatrix}\right)$ on $H^2(S_1; \mathbb{Z})$ is unimodular and even.)

Degree 3 follows from the perfect pairing $H^1 \times H^3 \to \mathbb{Z}$ and $\langle \pi_j^* a, \pi_j^* b \rangle = m_j \langle a, b \rangle$, the resulting index being read off from the determinants -3 and -2 of the pairing between V^{T_j} and $(\Lambda^3 V)^{T_j}$ recorded in Proposition 7.14; degree 4 follows from $\pi_{j*}[A_0^{(j)}] = m_j [S_j]$. ☐

Lemma A.5 (The mapping torus at p_0). Let $F = (\Lambda \otimes \mathbb{R})/\Lambda$, let Φ_0 be induced by M_0, and let $\mathcal{M}$ be its mapping torus, i.e. $f^{-1}(\partial)$ for a small circle ∂ around p_0. Then $H_*(\mathcal{M}; \mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}^3, \mathbb{Z}^6, \mathbb{Z}^6, \mathbb{Z}^3, \mathbb{Z})$, all free.

Proof. The Wang sequence of $\mathcal{M} \to S^1$ [Hat02] gives

$$0 \to \operatorname{coker}(\Lambda^k M_0 - I) \to H_k(\mathcal{M}; \mathbb{Z}) \to \ker(\Lambda^{k-1} M_0 - I) \to 0,$$

and by Lemma A.1(7) kernels and cokernels are free of ranks $(1, 2, 4, 2, 1)$, so the sequences split and the ranks add to $(1, 3, 6, 6, 3, 1)$. ☐

Remark A.6. Input to §7. Fix a radius ρ with $0 < \rho \leq \eta_0$ (recall that Lemma 7.5 provides $\eta_0 < \min(\epsilon, 1)$). The *closed* tube $N_0' := f_0^{-1}\{|t_c| \leq \rho/2\}$, of radius $\rho/2 \leq \eta_0$, retracts onto W by the retraction of §7.2 (Proposition 7.2), which is constructed Γ-equivariantly upstairs on $Y_{\leq \eta}$ and descended to the quotient; and $\partial N_0' \cong \mathcal{M}$. We do not assert a deformation retraction of the open tube N_0 onto W. For the *open* tube N_0 the argument used is of a different nature: the Step-1 flow of Lemma A.2(6) is available *upstairs only*, on Y, since it does not commute with the $\Psi_{\bar{\lambda}}$ and hence does not descend to the quotient. Downstairs we use instead the π_1 comparison: by Corollary 4.8 together with the simple connectivity of $t^{-1}(0)$,

$$\pi_1(N_0) = \overline{\Lambda} = \pi_1(W), \qquad H^1(N_0; \mathbb{Z}) \cong H^1(W; \mathbb{Z}).$$

Further: $N_j' = f_j^{-1}\{|t_j| \leq r'\}$ retracts onto S_j, with $\partial N_j'$ the mapping torus of $x \mapsto A_j x + v_j/m_j$ on F, whose m_j-fold fibrewise cover is $F \times S^1$; and $f^{-1}(B^\circ)$ retracts onto the F-bundle over a wedge of two circles given by A_1, A_2.

B A second proof of the invariants statement of Proposition 7.12.

Throughout this appendix we keep the notation of the local model at p_0: the proper holomorphic map $f_0 : N_0 \to \Delta_\varepsilon$ from the complex 3-manifold N_0, its central fibre $W = f_0^{-1}(0)$ (a reduced divisor), the double locus $\mathsf{D} = \bar{C}_1 \cup \bar{C}_2 \cup \bar{C}_3 \subset W$ with its two triple points P, Q, the smooth fibres $F = F_t = f_0^{-1}(t)$ for $0 < |t|$ small, the lattice $\Lambda = H_1(F; \mathbb{Z})$ and $V = H^1(F; \mathbb{Z})$, the monodromy M_0 on Λ (resp. T_0 on V), and the compact two-torus $T_f = \Lambda_{\mathrm{tor}} \otimes S^1 \subset T_c$ acting on Y_ε through the torus $T_{N'}$; this action preserves t and commutes with every $\Psi_{\bar{\lambda}} = (c_{\bar{\lambda}}(t), 1) \cdot \Phi_{\bar{\lambda}}$ (Lemma 4.3), because $T_{N'}$ is commutative and $\varphi_{\bar{\lambda}}$ restricts to the identity on T_f (Lemma 4.3(i): $\varphi_{\bar{\lambda}}$ is induced by the lattice map $(y, y_3) \mapsto (y + y_3 B_0 \bar{\lambda},\ y_3)$, which is the identity on the sublattice $y_3 = 0$ of cocharacters of T_f); hence T_f acts on $N_0 = Y_\varepsilon/\Gamma$ over Δ_ε, preserving f_0. We write $W^\circ := W \setminus \mathsf{D}$ and $\mathsf{D}^\circ := \mathsf{D} \setminus \{P, Q\}$, and $\nu : \mathfrak{F} \to W$ for the normalisation of W, where $\mathfrak{F}$ is the toric del Pezzo surface of degree 6 carrying the anticanonical hexagon $C_1, \dots, C_6$ of (-1)-curves (in cyclic order), ν identifying C_k with C_{k+3} to give $\bar{C}_k$, $k = 1, 2, 3$ (Theorem 4.5, Proposition 4.6).

The purpose of this appendix is to re-prove, by sheaf theory on W alone — without the retraction of N_0' onto W, without the collapse map and without the cell decomposition of §7.2 — the part of Proposition 7.12 which asserts that sp^q is injective with image the monodromy invariants.

B.1 Statement and strategy

Theorem B.1. Let $D \subset \Delta_\varepsilon$ be a small disc about 0 and $j\colon D^* \hookrightarrow D$ the inclusion of the punctured disc. For $0 \le q \le 4$ the constructible sheaf $R^q f_{0*}\mathbb{Z}_{N_0}$ satisfies

$$R^q f_{0*}\mathbb{Z} \xrightarrow{\cong} j_*j^*R^q f_{0*}\mathbb{Z} \qquad \text{on a neighbourhood of } 0 \in \Delta_\varepsilon;$$

equivalently, the specialisation map

$$\mathrm{sp}^q:\ H^q(W;\mathbb{Z}) = (R^q f_{0*}\mathbb{Z})_0 \longrightarrow (j_*j^*R^q f_{0*}\mathbb{Z})_0 = H^q(F;\mathbb{Z})^{T_0} \subset H^q(F;\mathbb{Z})$$

is an isomorphism onto the subgroup of monodromy invariants.

Remark B.2. The identifications in the statement are the standard ones: $(R^q f_{0*}\mathbb{Z})_0 = H^q(W;\mathbb{Z})$ by proper base change; $(j_*L)_0 = \varinjlim_r H^0(D_r^*, L)$ is the group of invariants of the local system $L = j^*R^q f_{0*}\mathbb{Z}$ with fibre $H^q(F;\mathbb{Z}) = \wedge^q V$ and monodromy $\wedge^q T_0$; and sp^q is

$$\varinjlim_r \Big[H^q\big(f_0^{-1}(D_r)\big) \longrightarrow H^q\big(f_0^{-1}(D_r^*)\big) \longrightarrow H^0(D_r^*, L) \Big].$$

In this formulation no homotopy equivalence $W \simeq N_0'$ is used. With it (Proposition 7.2), sp^q is also the restriction map $H^q(N_0') \to H^q(F)$, and Theorem B.1 is the invariants statement of Proposition 7.12.

Strategy. (i) Nearby cycles: the complex $\psi := \psi_{f_0}\mathbb{Z}_{N_0} \in D^b_c(W)$ computes $H^\bullet(F;\mathbb{Z})$, sp is the edge map of the hypercohomology spectral sequence

$$E_2^{a,b} = H^a(W; \mathcal{H}^b\psi) \Longrightarrow H^{a+b}(F;\mathbb{Z}),$$

and the cohomology sheaves $\mathcal{H}^b\psi$ are read off the three toric local models $t = z_0$, $t = z_0z_1$, $t = z_0z_1z_2$ of Theorem 4.5 (§B.2–§B.3). (ii) The E_2 page is computed in closed form (§B.4): rows $(1, 2, 4, 2, 1)$, $(2, 1, 3)$, (2), all free. (iii) Rank bookkeeping against $H^\bullet(F) = \wedge^\bullet\mathbb{Z}^4$ forces all differentials touching row 0 in degrees $\neq 3$ to vanish (§B.5). (iv) Degree 3 is settled by a compact-support (Gysin) argument along a single T_f-orbit in the smooth locus of W (§B.6). (v) Saturation and the identification with the invariants (§B.7).

The only inputs are: the standard facts on ψ_f for a proper holomorphic f [SGA7, Exp. XIII], [Dim04, §4.2], [KS90, §8.6]; the local models of Theorem 4.5; the Mayer–Vietoris resolution of $\mathbb{Z}_W$ attached to ν (recomputed in §B.4); exterior algebra of a rank-4 lattice; and the ranks $(1, 2, 4, 2, 1)$ of $(\wedge^q V)^{T_0} = \ker(\wedge^q T_0 - I)$, which are those of $\ker(\Lambda^q M_0 - I)$ by transposition (Lemma A.1(7); pure linear algebra from $T_0 = I + N$).

B.2 Nearby cycles for f_0

Let $\tilde{D}^* \to D^*$ be the universal cover, $k : N_0 \times_D \tilde{D}^* \to N_0$ the induced map, $i_W : W \hookrightarrow N_0$, and $\psi := i_W^* Rk_*k^*\mathbb{Z}_{N_0}$. We use the following standard facts [SGA7, Exp. XIII], [Dim04, §4.2], [KS90, §8.6].

(N1) $\psi \in D^b_c(W)$ and $\mathbb{H}^q(W;\psi) \cong H^q(F_t;\mathbb{Z})$ for $t \in D_r^*$ with r small, compatibly with the monodromy (f_0 is proper).

(N2) The unit $\mathbb{Z}_{N_0} \to Rk_*k^*\mathbb{Z}$ restricts to a map $\mathrm{sp} : \mathbb{Z}_W \to \psi$, and the induced map $H^q(W;\mathbb{Z}) \to \mathbb{H}^q(W;\psi) = H^q(F_t)$ is the specialisation map of Theorem B.1.

(N3) For $x \in W$ one has $(\mathcal{H}^b\psi)_x = H^b(\mathrm{MF}_x;\mathbb{Z})$, where $\mathrm{MF}_x := B_x \cap f_0^{-1}(t)$ is the Milnor fibre (B_x a small ball at x, $0 < |t|$ much smaller than its radius); and for $x' \in W \cap B_x$ close to x the restriction map $(\mathcal{H}^b\psi)_x \to (\mathcal{H}^b\psi)_{x'}$ of the constructible sheaf $\mathcal{H}^b\psi$ is induced by the inclusion $\mathrm{MF}_{x'} = B_{x'} \cap f_0^{-1}(t) \subset B_x \cap f_0^{-1}(t) = \mathrm{MF}_x$ (for $B_{x'} \subset B_x$).

Local models. By Theorem 4.5 (and Proposition 4.6), every $x \in W$ has a neighbourhood in N_0 biholomorphic over D to a neighbourhood of 0 in $\mathbb{C}^3$ in which

$$f_0 = z_0 \ \ (x \in W^\circ), \qquad f_0 = z_0 z_1 \ \ (x \in \mathsf{D}^\circ), \qquad f_0 = z_0 z_1 z_2 \ \ (x \in \{P, Q\}),$$

the coordinates being characters of the toric chart, so that the compact torus T_c acts diagonally by phases and $T_f \subset T_c$ is the subtorus preserving t.

Milnor fibres. For $f = z_0 \cdots z_k$ on $\mathbb{C}^{k+1} \times \mathbb{C}^{2-k}$ the Milnor fibre is $\{z_0 \cdots z_k = t,\ |z_i| < \rho\} \times (\text{polydisc})$; writing $z_i = r_i e^{\sqrt{-1}\theta_i}$, the radial part $\{\sum_i \log r_i = \log |t|,\ \log r_i < \log \rho\}$ is an open simplex, hence contractible, and the phase part is $\{\theta \in (S^1)^{k+1} :\ \sum_i \theta_i = \arg t\} \cong (S^1)^k$. Hence $\mathrm{MF}_x \simeq \mathrm{pt}$, S^1, T^2 for $x \in W^\circ$, D°, $\{P, Q\}$ respectively, and in the last two cases MF_x retracts onto an orbit of T_f: indeed T_f is the kernel of the character of T_c through which T_c acts on t, and it acts simply transitively on each phase torus $\{\sum_i \theta_i = \arg t\}$; for $k = 1$ the relevant orbit is that of the circle subgroup of T_f attached to the double curve through x. Therefore

$$\mathcal{H}^0\psi = \mathbb{Z}_W, \qquad \mathcal{H}^1\psi =: \mathcal{V} \text{ supported on } \mathsf{D}, \quad \operatorname{rk} \mathcal{V}_x = \begin{cases} 1, & x \in \mathsf{D}^\circ, \\ 2, & x \in \{P, Q\}, \end{cases}$$

$$\mathcal{H}^2\psi = \mathbb{Z}_P \oplus \mathbb{Z}_Q, \qquad \mathcal{H}^{\geq 3}\psi = 0.$$

(That $\mathcal{H}^0\psi = \mathbb{Z}_W$ is the statement that the Milnor fibres are connected and that $\mathbb{Z}_W \to \mathcal{H}^0\psi$ is stalkwise the isomorphism $\mathbb{Z} \to H^0(\mathrm{MF}_x)$.)

B.3 The sheaf $\mathcal{V} = \mathcal{H}^1\psi$

On D°. At a point of $\bar{C}_k^\circ$ the local model is $t = z_+ z_- \cdot (\text{unit})$ with the two local branches $W_\pm = \{z_\pm = 0\}$ of W; these are distinguished *globally* along $\bar{C}_k$ by the normalisation: W_+ is the branch which is the image under ν of a neighbourhood of $C_k \subset \mathfrak{F}$, and W_- that of C_{k+3}. Orient the vanishing circle $S = \{z_+ z_- = t'\}$ by $d \arg z_+ > 0$ (so $\arg z_-$ decreases along S). Since the rule is global along $\bar{C}_k$ and the Milnor fibration of $z_+ z_- = t'$ along the curve is locally trivial, $\mathcal{V}|_{\bar{C}_k^\circ}$ is the *trivial* rank-one local system $\mathbb{Z} \cdot [S]^\vee$. (Swapping $W_+ \leftrightarrow W_-$ reverses the orientation of S.)

At P. The model is $t = z_0 z_1 z_2$, with branches $W_i = \{z_i = 0\}$, $i = 0, 1, 2$, and double curves $L_{ij} = W_i \cap W_j$ (the z_l-axis, $\{i, j, l\} = \{0, 1, 2\}$). Here $\mathrm{MF}_P \simeq T^2$ with

$$H_1(\mathrm{MF}_P) = \{a \in \mathbb{Z}^3 :\ a_0 + a_1 + a_2 = 0\}, \qquad a \ \leftrightarrow\ \text{the loop } \theta \mapsto (\theta_i + a_i\theta)_{i=0,1,2},$$

and $\mathcal{V}_P = H^1(\mathrm{MF}_P) = \mathrm{Hom}(H_1(\mathrm{MF}_P), \mathbb{Z}) \cong \mathbb{Z}^2$. For $x' \in L_{ij}^\circ$ near P, the inclusion $\mathrm{MF}_{x'} \subset \mathrm{MF}_P$ is, up to homotopy, the circle along which θ_i increases, θ_j decreases and θ_l is constant (since $z_l \approx z_l(x') \neq 0$); oriented by "$\arg z_i$ increasing" its class is $e_i - e_j \in H_1(\mathrm{MF}_P)$, where e_0, e_1, e_2 is the standard basis of $\mathbb{Z}^3$. By (N3) the restriction $\mathcal{V}_P \to \mathcal{V}_{x'} \cong \mathbb{Z}$ is therefore $\varphi \mapsto \varphi(\pm(e_i - e_j))$, the sign being $+$ exactly when the global orientation rule on that curve is "$\arg z_i$ increasing".

At P the three global curves are $\bar{C}_1 = b_0 \cap b_1$, $\bar{C}_2 = b_0 \cap b_2$, $\bar{C}_3 = b_1 \cap b_2$, where b_0, b_1, b_2 are the three branches of W at P, with $C_1, C_2 \subset b_0$, $C_3, C_4 \subset b_1$, $C_5, C_6 \subset b_2$ (combinatorics of ν on the hexagon). With the global rule above (W_+ the C_k-branch, orientation $d \arg z_+ > 0$, z_+ the equation of W_+) and $b_i = W_i = \{z_i = 0\}$, the ordered pairs (W_+, W_-) at P are

$$\bar{C}_1 : (b_0, b_1), \qquad \bar{C}_2 : (b_0, b_2), \qquad \bar{C}_3 : (b_1, b_2),$$

so that the three restriction maps are

$$\varphi \longmapsto (n_1, n_2, n_3) = \big(\varphi(e_0 - e_1),\ \varphi(e_0 - e_2),\ \varphi(e_1 - e_2)\big), \qquad \text{whence} \quad n_1 - n_2 + n_3 = 0.$$

Conversely every $(n_1, n_2, n_3) \in \mathbb{Z}^3$ with $n_1 - n_2 + n_3 = 0$ arises from a unique φ, since $e_0 - e_1$, $e_1 - e_2$ is a basis of $H_1(\mathrm{MF}_P)$.

At Q the branches are $c_0 \supset C_2, C_3$, $c_1 \supset C_4, C_5$, $c_2 \supset C_6, C_1$, and the ordered pairs are $\bar{C}_1 : (c_2, c_1)$, $\bar{C}_2 : (c_0, c_1)$, $\bar{C}_3 : (c_0, c_2)$, giving

$$(n_1, n_2, n_3) = \big(\varphi(e_2 - e_1),\ \varphi(e_0 - e_1),\ \varphi(e_0 - e_2)\big)$$

and again $n_1 - n_2 + n_3 = 0$: the *same* linear form as at P (this is the sign check; with the opposite uniform rule all three n_k change sign at both points, and the relation is unchanged).

Conclusion. $\mathcal{V}$ is a subsheaf of $\bigoplus_{k=1}^{3} \iota_{k*}\mathbb{Z}_{\bar{C}_k}$ via restriction to the branches of D (injective at P and Q by the basis statement, an isomorphism on D°), with cokernel supported at $\{P, Q\}$:

$$0 \longrightarrow \mathcal{V} \longrightarrow \bigoplus_{k=1}^{3} \iota_{k*}\mathbb{Z}_{\bar{C}_k} \xrightarrow{\delta_1} \mathbb{Z}_P \oplus \mathbb{Z}_Q \longrightarrow 0, \qquad \delta_1(n) = \big((n_1 - n_2 + n_3)(P),\ (n_1 - n_2 + n_3)(Q)\big). \quad \text{(B.1)}$$

(Exactness at P: sections of $\bigoplus_k \iota_{k*}\mathbb{Z}$ on a small neighbourhood of P are triples of integers, those coming from $\mathcal{V}_P$ are exactly $\ker \delta_1$, and δ_1 is onto; similarly at Q.) This is the same map δ_1 as in the Mayer–Vietoris resolution of $\mathbb{Z}_W$ used below.

B.4 The E_2 page

We compute $E_2^{a,b} = H^a(W; \mathcal{H}^b\psi) \Rightarrow H^{a+b}(F; \mathbb{Z})$, with $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$.

Row $b = 0$: $E_2^{a,0} = H^a(W; \mathbb{Z})$. We use the Mayer–Vietoris resolution attached to ν,

$$0 \to \mathbb{Z}_W \to \nu_*\mathbb{Z}_{\mathfrak{F}} \to \bigoplus_{k=1}^{3} \iota_{k*}\mathbb{Z}_{\bar{C}_k} \xrightarrow{\delta_1} \mathbb{Z}_P \oplus \mathbb{Z}_Q \to 0, \qquad \text{(B.2)}$$

whose exactness is checked by the same local computations as for (B.1) (the first map after $\mathbb{Z}_W$ being the difference of the restrictions to C_k and to C_{k+3}). With $H^\bullet(\mathfrak{F}) = (\mathbb{Z}, 0, \mathbb{Z}^4, 0, \mathbb{Z})$ and $H^\bullet(\mathbb{P}^1) = (\mathbb{Z}, 0, \mathbb{Z})$, the E_1 page of the hypercohomology spectral sequence of (B.2) has

$$\text{row } 0: \quad \mathbb{Z} \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{(1,1)^t(1,-1,1)} \mathbb{Z}^2 \qquad \text{(cohomology } \mathbb{Z},\ \mathbb{Z}^2,\ \mathbb{Z} \text{ in degrees } 0, 1, 2),$$

$$\text{row } 2: \quad H^2(\mathfrak{F}) = \mathbb{Z}^4 \xrightarrow{d} \bigoplus_k H^2(\bar{C}_k) = \mathbb{Z}^3 \to 0, \qquad d(L) = (L \cdot C_k - L \cdot C_{k+3})_{k=1,2,3},$$

$$\text{row } 4: \quad \mathbb{Z}.$$

In $\operatorname{Pic}\mathfrak{F} = \langle H, E_1, E_2, E_3 \rangle$ the hexagon is $E_1,\ H - E_1 - E_2,\ E_2,\ H - E_2 - E_3,\ E_3,\ H - E_1 - E_3$ in cyclic order, with opposite pairs $(E_1, H - E_2 - E_3)$, $(H - E_1 - E_2, E_3)$, $(E_2, H - E_1 - E_3)$; hence for $L = aH - \sum_i b_i E_i$ one finds $d(L) = (s, -s, s)$ with $s = b_1 + b_2 + b_3 - a$. So $\operatorname{rk} d = 1$, $\operatorname{im} d = \mathbb{Z}\,(1, -1, 1)$ is saturated, $\ker d \cong \mathbb{Z}^3$ and $\operatorname{coker} d \cong \mathbb{Z}^2$ is free. All E_1-rows of odd index vanish, so this spectral sequence degenerates at E_2 and

$$H^\bullet(W; \mathbb{Z}) = \big(\mathbb{Z},\ \mathbb{Z}^2,\ \mathbb{Z} \oplus \ker d \cong \mathbb{Z}^4,\ \operatorname{coker} d \cong \mathbb{Z}^2,\ \mathbb{Z}\big), \qquad \text{all free}$$

(extensions of free groups by free groups). This recovers the ranks $(1, 2, 4, 2, 1)$ of $H^\bullet(W; \mathbb{Z})$ of Proposition 7.11, though not its description of generators.

Row $b = 1$: $E_2^{a,1} = H^a(W; \mathcal{V}) = H^a(\mathsf{D}; \mathcal{V})$. From (B.1) and $H^\bullet(\bar{C}_k) = (\mathbb{Z}, 0, \mathbb{Z})$ we get

$$0 \to H^0(\mathcal{V}) \to \mathbb{Z}^3 \xrightarrow{\delta_1} \mathbb{Z}^2 \to H^1(\mathcal{V}) \to 0, \qquad H^2(\mathcal{V}) \cong \bigoplus_k H^2(\bar{C}_k) = \mathbb{Z}^3, \qquad H^{\geq 3}(\mathcal{V}) = 0.$$

As $\delta_1(n) = (s, s)$ with $s = n_1 - n_2 + n_3$, the image is the saturated subgroup $\mathbb{Z}\,(1, 1)$, whence

$$H^\bullet(W; \mathcal{V}) = \big(\mathbb{Z}^2,\ \mathbb{Z},\ \mathbb{Z}^3,\ 0\big), \qquad \text{all free.}$$

Row $b = 2$: $E_2^{0,2} = H^0(W; \mathbb{Z}_P \oplus \mathbb{Z}_Q) = \mathbb{Z}^2$ and $E_2^{a,2} = 0$ for $a > 0$.

So the E_2 page (ranks; all groups free) is

	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$
$b=2$	2	0	0	0	0
$b=1$	2	1	3	0	0
$b=0$	1	2	4	2	1

total 18, $\quad \sum_q \operatorname{rk} H^q(F) = \sum_q \binom{4}{q} = 16.$

B.5 Differentials: everything touching row 0 vanishes, except possibly in total degree 3

All $E_2^{a,b}$ are free (§B.4). We first track ranks; saturation of the images is dealt with in §B.7.

Since $H^\bullet(F; \mathbb{Z}) = \wedge^\bullet \mathbb{Z}^4$ is free of ranks $(1, 4, 6, 4, 1)$, the ranks of the E_∞ terms in total degree q sum to

$\binom{4}{q}$. By position, the only possibly non-zero differentials are

$$\alpha := \mathrm{rk}\left(d_2 : E_2^{0,1} \to E_2^{2,0}\right), \quad \beta := \mathrm{rk}\left(d_2 : E_2^{1,1} \to E_2^{3,0}\right), \quad \gamma := \mathrm{rk}\left(d_2 : E_2^{2,1} \to E_2^{4,0}\right),$$
$$\delta := \mathrm{rk}\left(d_2 : E_2^{0,2} \to E_2^{2,1}\right), \qquad \epsilon := \mathrm{rk}\left(d_3 : E_3^{0,2} \to E_3^{3,0}\right);$$

all other d_r have source or target 0 (e.g. $E_2^{1,2} = 0$, and $E_2^{a,b} = 0$ for $b \geq 3$ or for $a \geq 3,\ b \geq 1$). Total-degree bookkeeping gives:

- degree 1: $4 = \mathrm{rk}\, E_\infty^{1,0} + \mathrm{rk}\, E_\infty^{0,1} = 2 + (2 - \alpha)$, whence $\alpha = 0$;
- degree 4: $1 = \mathrm{rk}\, E_\infty^{4,0} = 1 - \gamma$, whence $\gamma = 0$;
- degree 2: $6 = (4 - \alpha) + (1 - \beta) + (2 - \delta - \epsilon) = 7 - \beta - \delta - \epsilon$, whence $\beta + \delta + \epsilon = 1$;
- degree 3: $4 = (2 - \beta - \epsilon) + (3 - \gamma - \delta)$, whence $\beta + \gamma + \delta + \epsilon = 1$ (consistent).

Hence $E_\infty^{q,0} = E_2^{q,0} = H^q(W;\mathbb{Z})$ for $q = 0, 1, 2, 4$; that is, sp^q is injective for $q \neq 3$ (the edge map $E_2^{q,0} \twoheadrightarrow E_\infty^{q,0} \hookrightarrow H^q(F)$ is sp^q by (N2)), and exactly one of β, δ, ϵ equals 1.

B.6 Degree 3: sp^3 has rank 2

Lemma B.3 (specialisation with supports)**.** Let $\mathcal{K} \subset W^\circ$ be compact. There are an open neighbourhood K' of $\mathcal{K}$ in W°, a radius $r > 0$, and an open embedding $\Theta : K' \times D_r \hookrightarrow N_0$ over D_r with $\Theta(\cdot, 0) = \mathrm{id}_{K'}$; if $\mathcal{K}$ is T_f-invariant, K' and Θ can be taken T_f-equivariant. For $t \in D_r^*$ put $\mathcal{K}_t := \Theta(\mathcal{K} \times \{t\}) \subset F_t$. Then the composition

$$H^q(W, W \smallsetminus \mathcal{K}) \longrightarrow H^q(W) \xrightarrow{\mathrm{sp}^q} H^q(F_t)$$

equals

$$H^q(W, W \smallsetminus \mathcal{K}) \ \cong\ H^q(F_t, F_t \smallsetminus \mathcal{K}_t) \longrightarrow H^q(F_t),$$

the isomorphism being Θ-transport.

Proof. Since f_0 is a submersion at every point of the compact set $\mathcal{K} \subset W^\circ$ (on W° the local model is $f_0 = z_0$), there is a relatively compact open neighbourhood K'' of $\mathcal{K}$ in N_0 on which f_0 is a submersion; replacing K'' by $T_f \cdot K''$ — still open, relatively compact, and contained in the locus where f_0 is a submersion, which is T_f-invariant because T_f preserves f_0 — we may assume K'' is T_f-invariant; choose a Riemannian metric on K'', averaged over the compact group T_f (which preserves f_0), and let ξ_1^h, ξ_2^h be the horizontal lifts (orthogonal to $\ker df_0$) of $\partial_{\mathrm{Re}\, t}, \partial_{\mathrm{Im}\, t}$. For $t \in \mathbb{C}$ put

$$\xi_t := \mathrm{Re}(t)\, \xi_1^h + \mathrm{Im}(t)\, \xi_2^h,$$

a single vector field depending linearly on the parameter t, with $df_0(\xi_t) \equiv t$. For $r > 0$ small the time-one flow $\Phi_1^{\xi_t}$ is defined on a neighbourhood $K' \subset W^\circ$ of $\mathcal{K}$ for all $|t| < r$, and we set $\Theta(w, t) := \Phi_1^{\xi_t}(w)$. Then $f_0(\Theta(w,t)) = t$ (as $f_0(w) = 0$ and $\frac{d}{ds} f_0 = t$ along the flow), $\Theta(\cdot, 0) = \mathrm{id}$, Θ is smooth in (w, t), its differential at $K' \times \{0\}$ is bijective (the identity on $T_w W^\circ$, and the ξ-directions onto $T_0 D$), and Θ is injective ($\Theta(w,t) = \Theta(w',t')$ forces $t = t'$ by applying f_0, and then $w = w'$ by injectivity of a time-one flow); shrinking K' and r, the map $\Theta : K' \times D_r \hookrightarrow N_0$ is therefore an open embedding over D_r. If $\mathcal{K}$ is T_f-invariant, take K' to be T_f-invariant; since the metric, hence ξ_t, is T_f-invariant, Θ is T_f-equivariant.

Let now $A_r := f_0^{-1}(D_r)$ and $Z := \Theta(\mathcal{K} \times D_r)$, which is closed in A_r. The statement at the level of classes follows from the commutative diagram of restriction maps of pairs

$$H^q(A_r, A_r \smallsetminus Z) \to H^q(A_r) \to H^q(F_t),$$
$$H^q(A_r, A_r \smallsetminus Z) \to H^q(W, W \smallsetminus \mathcal{K}),$$
$$H^q(A_r, A_r \smallsetminus Z) \to H^q(F_t, F_t \smallsetminus \mathcal{K}_t),$$

in which the last two maps are isomorphisms: by excision both are computed by

$$H^q\big(\Theta(K' \times D_r),\ \Theta((K' \smallsetminus \mathcal{K}) \times D_r)\big) \cong H^q\big(K' \times D_r,\ (K' \smallsetminus \mathcal{K}) \times D_r\big),$$

which restricts isomorphically to the slices $t' = 0$ and $t' = t$ by homotopy invariance. Finally, sp^q is $\varinjlim_r \left[H^q(A_r) \to H^q(F_t)\right]$ precomposed with the inverse of the isomorphism $\varinjlim_r H^q(A_r) \xrightarrow{\cong} H^q(W)$ (proper base change), and the classes we feed in come from $H^q(A_r, A_r \smallsetminus Z)$ for every small r, compatibly. $\square$

Application. Let $\mathcal{K} := T_f \cdot w_0 \subset W^\circ$ be a T_f-orbit, w_0 a point of the open orbit $W^\circ \cong (\mathbb{C}^*)^2$ of the toric surface W; there T_f acts freely, so $\mathcal{K} \cong T^2$, with trivial (T_f-equivariantly trivial) normal bundle in the 4-manifold W°. Then $\mathcal{K}_t = T_f \cdot \Theta(w_0, t)$ is a T_f-orbit in F_t, i.e. a translate of the linear 2-subtorus of $F_t = (\Lambda \otimes \mathbb{R})/\Lambda$ spanned by Λ_{tor} (on $F_t = Y_t/\Gamma$ the compact torus $T_f \subset (\mathbb{C}^*)^2 = Y_t$ acts by multiplication, i.e. by translations along $\Lambda_{\mathrm{tor}} \otimes \mathbb{R}$ in the flat coordinates of Proposition 4.7, whose marking identifies $H_1((\mathbb{C}^*)^2)$ with Λ_{tor}; the action is free and its orbits are the translates of $(\Lambda_{\mathrm{tor}} \otimes \mathbb{R})/\Lambda_{\mathrm{tor}}$), of class $\hat{w} \wedge \hat{\delta} \in H_2(F)$.

By the Thom isomorphism for the trivial normal bundle,

$$H^3(W, W \smallsetminus \mathcal{K}) \cong H^1(\mathcal{K}) \otimes H^2(D^2, D^2 \smallsetminus 0) \cong \mathbb{Z}^2,$$

and likewise in F_t, and $H^3(F_t, F_t \smallsetminus \mathcal{K}_t) \to H^3(F_t)$ is the Gysin map $\iota_! : H^1(\mathcal{K}_t) \to H^3(F_t)$ of the subtorus. For a linear subtorus $T_f \subset F = T_f \times T_B$ (any complementary linear subtorus T_B; topologically $F_t \to F_t/T_f$ is a trivial principal T_f-bundle, Λ_{tor} being saturated in Λ, so that $\Lambda = \Lambda_{\mathrm{tor}} \oplus \Lambda'$ for some Λ' and $F_t \cong T_f \times T_B$ with $T_B := (\Lambda' \otimes \mathbb{R})/\Lambda'$) one has

$$\iota_!(a) = a \times [T_B]^\vee,$$

which is injective, $H^1(T_f) \hookrightarrow H^1(T_f) \otimes H^2(T_B) \subset H^3(F)$, with *saturated* image (a Künneth direct summand).

By Lemma B.3, $\mathrm{sp}^3 \circ \left(H^3(W, W \smallsetminus \mathcal{K}) \to H^3(W)\right) = \iota_! \circ (\Theta\text{-transport})$ has rank 2. Hence $\mathrm{rk}\,\mathrm{sp}^3 \geq 2 = \mathrm{rk}\, H^3(W)$, so sp^3 is injective ($H^3(W)$ being free); equivalently $\mathrm{rk}\, E_\infty^{3,0} = 2$, i.e. $\beta = \epsilon = 0$, and therefore $\delta = 1$ by §B.5. Moreover $\mathrm{im}\,\mathrm{sp}^3 \supseteq \mathrm{im}\,\iota_!$, both of rank 2 and the latter saturated; so $\mathrm{im}\,\mathrm{sp}^3 = \mathrm{im}\,\iota_!$ is saturated.

B.7 Saturation and invariants

- $q = 0$: trivial. $q = 4$: $E_\infty^{4,0} = H^4(F)$ (the other pieces $E_\infty^{3,1}, E_\infty^{2,2}$ vanish), so sp^4 is an isomorphism.
- $q = 1$: $H^1(F)/\mathrm{im}\,\mathrm{sp}^1 = E_\infty^{0,1} = H^0(W; \mathcal{V}) \cong \mathbb{Z}^2$ is free ($\alpha = 0$), so $\mathrm{im}\,\mathrm{sp}^1$ is saturated of rank 2.
- $q = 2$: $H^2(F)/F^2$ has graded pieces $E_\infty^{1,1} = H^1(W; \mathcal{V}) \cong \mathbb{Z}$ (as $\beta = 0$) and $E_\infty^{0,2} = \ker\left(d_2|_{E_2^{0,2}}\right) \subset \mathbb{Z}^2$, both free; hence $\mathrm{im}\,\mathrm{sp}^2 = F^2H^2(F)$ is saturated of rank 4.
- $q = 3$: §B.6.

In each degree $\mathrm{im}\,\mathrm{sp}^q \subseteq H^q(F)^{T_0}$ (classes extending over $f_0^{-1}(D_r)$ are monodromy invariant), $\mathrm{im}\,\mathrm{sp}^q$ is saturated of rank $b_q(W)$, and the ranks $b_\bullet(W) = (1, 2, 4, 2, 1)$ agree with

$$\mathrm{rk}\, H^q(F)^{T_0} = \mathrm{rk}(\wedge^q V)^{T_0} = (1, 2, 4, 2, 1)$$

(Lemma A.1(7)). A saturated subgroup of the (saturated) invariant subgroup, of the same rank, is the whole of it. This proves Theorem B.1. $\square$

B.8 Scope

Remark B.4. Theorem B.1 supplies the injectivity of sp^q and the identification of its image with the T_0-invariants, which is what §§7.4–7.7 use of Proposition 7.12; the identification of sp^q with c^* for the collapse map c, the generators of $H_2(W)$, and the normalisations $\mathrm{sp}^2(\mathbb{T}_B^\vee) = \gamma \wedge u$, $\mathrm{sp}^2(\bar{C}_k^\vee)$ of Proposition 7.12 are not re-proved here and continue to rest on §7.2 and Proposition 7.12. The ancillary facts used above — the toric local models of Theorem 4.5, the hexagon combinatorics of Proposition 4.6, the flat model of $F_t = Y_t/\Gamma$ with its T_f-action (Proposition 4.7), and Lemma A.1 — do not involve §7.2; the ranks $(1, 2, 4, 2, 1)$ of $H^\bullet(W; \mathbb{Z})$ are computed in §B.4 from the resolution (B.2) and agree with, but are not taken from, Proposition 7.11.

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